Lecture 7: Optimization, Equilibrium (how odd), and Complexity

A foolish consistency is the hobgoblin of simple minds - *R.W. Emerson*

1 Introduction

In this lecture, we explore the link between assumptions about human behavior and the aggregative conclusions. When we "prove" a mathematical result, we are demonstrating what follows from our assumptions. Fermat's theorem and the Pythagorean theorem are true once you make the core assumptions of Euclidean geometry. They just may not have been obvious prior to their proof. Their veracity is sort of amazing, if you are willing to accept that math can be cool.

In the first lecture, we discussed the possible assumptions we might make about human behavior in political and economic contexts. One of the standard assumptions is that people optimize. This assumption reduces the set of possible outcomes that a model can produce. In this lecture, we will see that he optimization assumption often leads to the conclusion that the model produces equilibria (unfortunately, the reasons are rather complicated). Generating equilibria is a good thing provided that these equilibria help us to predict, predict other things, explain, design, and produce insights.

Note that the converse of this claim does not hold. Equilibrium does not imply that the agents within the model are optimizing. If everyone does the same stupid thing every day, they will be in equilibrium but they will not be optimizing. So the set of models which generate equilibria contains all optimization models (given a few technical caveats) and many other models.

As we have already discussed, the notion of the world being in equilibrium should strike us as strange. If we look about us at our social world, we see many transient phenomena. Why would anyone think of the world as being in equilibrium? That is a good question and it is one that we will answer.

This lecture has six parts. First, we construct a simple model that generates an equilibrium. Second, we then discuss some simple mathematics to explain why. Third, we then turn to some amazing mathematics that will seem unrelated to anything we care about. We take this mathematics and apply it models of human behavior using a very neat trick. We use this trick to explain why the optimization assumption gets us to equilibrium. We conclude by examining whether this restriction to equilibrium rests on solid footing and by considering a specific policy.

2 A model of equilibrium

Near UM's campus there are approximately sixty places to grab lunch. On each Monday, assume that the same set of 3000 people go out for lunch each day. Each of these people has a preference for each of the 60 restaurants given by an integer from 0 to 100, where 100 means that a person really likes a restaurant and zero means that the person hates it. Let's index the people from one to three thousand and the restaurants from one to sixty. Let person i's preference for restaurant j equal X_{ij}. This preference takes into account the quality and price of the food and the ambiance, but it does not take into account the distance between where the person works and the restaurant. Nor does it take into account the time spent waiting for a table. Assume that person i has a preference for time α_i. The larger the α_i, the more impatient a person is. Let w_{ij} equal the time it takes for person i to walk to restaurant j and let T_j be the time spent waiting at restaurant j, This is a function of the number of people who choose j, which we denote by N_j. With all of this information, we can then write person i's total utility from going to restaurant i as

$$U_{ij} = X_{ij} - \alpha_i[w_{ij} + T_j(N_j)]$$

To start this system running, assume that initially each person assume that $T_j(N_j)$ will be zero and goes to the restaurant that gives him or her the highest utility. If preferences are correlated (if people like the same restaurants and work in similar places) , then some restaurants will be crowded. The next day, some people will want to choose different restaurants. If we assume that only the most unhappy person changes his or her choice each day, then we can show that this system will settle into an equilibrium. To show this, we rely on a theorem that sounds difficult but is quite easy.

3 An obvious theorem with an imposing name

The obvious theorem relies on something called an energy or Lyaponuv function. The idea is this. Suppose that I have a process that occurs each period, like people choosing a restaurant. Further suppose I can write a function F that describes that process so that it satisfies two properties (i) if the process is not at an equilibrium, F increases by at least some amount Δ (set $\Delta = 1$ because it makes everything nice) and (ii) that F has a maximum value, say 1000. We can then write a theorem:

Theorem 1 *If we can write an F (a Lyaponuv Function) satisfying (i) and (ii), then the system goes to an equilibrium.*

Why? If the value of F equals 487 and the system is not at an equilibrium, then the next period F has to have a value of at least 488. Since the biggest F can be is 1000, this process has to stop.

Now, let's try to apply framework this to our lunch model. Let F be the sum of how happy everyone is. Now think about what happens when a person moves t a new restaurant. There are three effects. First, the person becomes happier. He makes everyone at the crowded restaurant he left happier and everyone at the new restaurant a little less happy. If on average people leave crowded restaurants for less crowded (this may take some work to show mathematically), then it would be the case that aggregate utility goes up by some fixed amount each time someone moves. Therefore, the process stops.

There is a nagging question of what does this all mean. How do we apply it? Lyaponuv framework applies to situations in which there is a single equilibrium and in which there is a force toward that equilibrium. there is a tired phrase that some actions are "as easy as falling down a hill." That is what the Lyaponuv framework captures: situations in which nothing can stop the equilibrium from happening. So, when would this apply in the world of humans (as opposed to in the world of mathematics)? One important case is markets. It is possible to construct Lyaponuv functions for exchange markets. This means that there are conceptualizations of the supply equals demand equilibrium that present the equilibrium as inevitable. This suggests that market equilibrium makes sense and will probably occur.

4 An Amazing Theorem

Lyaponuv Functions are one reason why systems of interacting people settle into equilibria, but the more compelling reason is explained by something called Brouwer's Fixed Point theorem. This is the theorem that John Nash and John Von Neumann and Ken Arrow exploited to prove that games have equilibria and that the economy does.

4.1 The mathematics

Suppose that I take a glass of water and swirl it around smoothly. After I am finished, there is at least one molecule in *exactly* the same place that it started. This is Brouwer's Fixed Point theorem. We can write this formally as

Theorem 2 *Any continuous function F which maps a connected, topologically ("rubber") convex set into itself has a fixed point.*

There some technical things to flesh out here. Continuous means you can't pick up the pencil. Connected means you can get from any one point to any other without leaving the set. Convex means that if you draw the shortest line between the points, then you don't ever leave the set. A circle is convex. A donut is not. Topological convexity means that you can start with a convex set and stretch it but not poke any holes in it. A liver shape is topologically convex because it is just a contorted circle.

Let's walk through an example. Suppose I map an orange to itself. If I spin it, the fixed points are the axis or rotation. If I spin it on another axis, the only fixed point is the center. If I then push on the outside until I displace the center, Brouwer's fixed point theorem implies that I am simultaneously creating other fixed points. What is happening is that points get displaced from their locations.

This theorem is deep and extremely hard to prove. In one dimension though it is easy. If F continuously maps the interval $[0, 1]$ into itself, then there has to be a fixed point because $F[0]$ must be above 0 (otherwise 0 is a fixed point) and $F[1]$ must be below 1 (otherwise 1 is a fixed point). If we then try to connect those two points, we must cross the 45 degree line, which is the set of fixed points.

4.2 Even more amazing weird math

In the one dimensional version of the fixed point theorem, our pencil must generically cross the 45 degree line an odd number of times. Metaphorically: if I start on one side of the river and end on the other, I've crossed the river an odd number of times. This means that the number of fixed points in one dimension must be odd except in rare cases. In the mathematics, the line is infinitely narrow so if I dip my foot in and turn around, then I'm "crossing" it. This is why it is possible to have an even number of equilibria. Amazingly, that is true in higher dimensions as well.

Theorem 3 *Any continuous function F which maps a connected, topologically ("rubber") convex set into itself has an odd number of fixed points with probability one.*

We say that this occurs with probability one. This means that it could fail to be true if you contrived an example but generically it would not.

5 Applying this theorem

What does this math have to do with anything we care about? Well, with some cobbling, we can use this theorem to show that fixed point are equilibria. Let's suppose that there are two people, you and me, and my optimal action depends on your action and your optimal action depends on my action. We can turn this verbal description into mathematics:

$$action_{me} = H(action_{you})$$

$$action_{you} = G(action_{me})$$

Suppose that our action spaces and the functions H and G satisfy all of the assumptions of the theorem. (This is not an innocuous assumption, we typically have to do some fudging and rely on an extension of Brouwer's theorem called Kakutani's fixed point theorem, but that just gets too mathematical.) We can now write a function F as follows:

$$F(action_{me}, action_{you}) = (H(action_{you}), G(action_{me}))$$

a fixed point of this map has two properties.

(i) you are optimizing given my action

I am optimizing given your action.

This is an equilibrium. No one has any incentive to change what they are doing.

What we need to do now is show that this situation fits the assumptions of he theorem. First, notice that the only place that optimality enters the analysis is in the H and G functions. If we are taking optimally behaviors then it is more likely that our actions are continuous (if we behaved randomly,then these functions might not be nice and the theorem wouldn't hold.) We can capture this logic in a theorem.

Theorem 4 *If the space of actions is "nice" and the responses by people are continuous, then there exists an equilibrium.*

This equilibrium need not be unique, but typically there will be an odd number of them. This theorem is why economists and many political scientists use equilibrium models.

6 Do we buy this?

If we buy into this mathematical story, then the world goes to equilibrium. But, yet this seems at odds with reality. To bridge this disconnect, we can make several observations about why real world systems may not settle into equilibria. The first set of observations correspond to attributes of the system. The initial quote by Emerson corresponds to problems #2 and #4.

Problem #1 Big spaces Just because there exists an equilibrium doesn't mean that society will obtain it on a time scale that matters. The second law of thermodynamics says that we'll all die of heat death as randomness grows. The equilibrium is not that relevant for deciding on lunch. So, if the system is huge, if it has lots of parts and is slow moving, equilibrium may not occur. That could be the case in the economy writ large.

Problem # 2 Entrants and Exits The theorem takes the number of players as given. If players enter and leave the game then we could be heading toward a moving target. It is also possible that people change over time. In effect, I may become a new person every day.

Problem #3 External Shocks and Feedbacks When we construct a model, we isolate it from everything else. It may well be that the domain we are studying is influenced by other domains, that it is part of a big systems. Therefore, if there is a shock to one part of the system it may ripple through all the other layers and create elaborate feedbacks.

The next set of observations correspond to assumptions about behavior.

Problem # 4 New Actions The set of possible actions could change over time (possibly through technology) or be so large that the "action" takes place on only a small subset at any given time.

Problem # 5 Low Stakes: Rules of Thumb The stakes may be low and so people may use rules of thumb and not optimize. I follow a rule of thumb when deciding what to wear. These rules of thumb may be discontinuous. For example, they could be of the form that you make a big change if some event occurs. Of course, it is possible that systems with rules of thumb equilibrate, but they do not have to do so.

Problem #6 Hard Problem: Rules of Thumb The problem may be hard (like chess) or information may be bad and so rather than optimize people may be forced to rely on rules of thumb.

Problem #7 Complex Dynamics Lyaponuv functions are like giant vacuums. The system gets sucked into the equilibrium. That is not true in general. There need not be anything that forces the systems to an equilibrium. If people best respond to the one another, we could cycle or get chaos. The existence of an equilibrium doesn't mean that we can come up with a rule that gets us to it. It just means that once we are at the equilibrium we will stay there. This is what makes modeling hard. We need to have a good understanding of how people behave in order to understand whether they will attain an equilibrium.

7 Why do we care?

We construct models to predict, design, understand, etc.... If we accept the equilibrium paradigm then it simplifies the processes of model building and analyzing. Let's consider school voucher proposals. An equilibrium view would say that all we need to do is compare the equilibrium with vouchers to the equilibrium without vouchers. If the former is better (and it probably is), then we should move to a voucher system. The problem is that if *either* the current system or the voucher systems is not going to be in equilibrium, that comparison is not relevant. It is a comparison of ideals in a world of imperfections. If we consider our eight reasons why something may not be in equilibrium, we can ask whether the current system or a voucher systems might suffer from that problem.

Problem	*Current System*	*School Vouchers*
Too Many People and Actions	No	Maybe
Entrants and Exits	Yes	Yes
System Feedbacks	Yes	Yes
New Actions	No	No
Low Stakes: ROT	Yes	Yes
Too Hard: ROT	Yes	Yes
Complex Dynamics	No	Maybe