A Pricing Mechanism which Implements a Network Rate Allocation Problem in Nash Equilibria

Tudor M. Stoenescu and John O. Ledyard

***Abstract*— We present a novel pricing mechanism which achieves a solution to a rate allocation problem in unicast service provisioning. This mechanism is different from the ones which appear in the existing literature because it takes into account the strategic behavior of individual users. The main features of this mechanism are: (i) it implements the rate allocation problem in Nash equilibria; (ii) it is budget balanced; and (iii) it satisfies voluntary participation. We provide some insight on how one may generalize this mechanism, and we determine a particular network structure under which this mechanism is informationally efficient.**

***Index Terms*— Pricing mechanisms in networks, Nash implementation, utility maximization, informational efficiency.**

I. Introduction

In today's fast paced world communication networks are designed to support the delivery of a variety of services to their users. Due to high user demand and finite network resources, one of the major challenges in the operation of such networks is the design of efficient resource allocation schemes which maximize the network's usefulness for its users. The challenge in determining such resource allocation schemes comes from the fact that the network problem is informationally decentralized. The informational constraints imposed by the nature of the network problem can be described as follows: (i) Each user has preferences over the set of services offered by the network. User preferences are generally characterized by a utility function, which is private information and is not known to the network or the other users; (ii) Each user is not directly aware of the other users requesting services from the network; and (iii) the network (network manager) knows the network topology and the network's resources (e.g. link capacities, buffer sizes), but is unaware of the number of users that may request services, as well as of the users' utilities.

Decentralized resource allocation problems have been explored in great detail by mathematical economists in the context of mechanism design. Mechanism design consists of two components: realization theory and implementation theory. Realization theory deals with the design of a set of rules which: (i) satisfy the informational constraints of the allocation problem; and (ii) are such that, if followed, generate allocations that satisfy a social choice rule/goal correspondence. These rules do not take into account issues of strategic behavior of individual agents who are involved in the allocation process. Implementation theory is concerned with strategic behavior of allocation processes. That is, it is concerned with the design of "game forms" that implement in some behavioral equilibrium (solution concept) social choice rules.

Within the context of communication networks, most of the existing literature has approached the design of resource allocation mechanisms from the point of view of realization theory. The predominant type of mechanism that has been studied is a pricing mechanism. Most of these have been inspired by a mechanism proposed by Kelly [8]. The main features of Kelly's model are as follows: (1) the network is assumed to own resources which are bundled together in the form of services, and are allocated to the users; (ii) the network sets up resource prices which in turn generate service prices; (iii) each user uses his service price to calculate an allocation that would maximize his net utility. The goal of the mechanism is to find prices and allocations that clear this pseudo-market. The key assumption that allows the mechanism to succeed is that users behave as "price takers" (that is, they do not anticipate the effects of their actions and reports on the "market clearing" prices and allocations).

A user is said to behave "strategically" if he does not behave as a price taker and exploits the fact that his actions/demands affect the network's service prices and his own allocation. In [4, 6] the authors show that in the case in which the users behave strategically, the resource allocations generated by Kelly's mechanism[1] suffer from a certain "efficiency loss".[2] Particularly, in the case in which each user bids on individual network resources, [6] shows that there exists a lower bound on the efficiency loss. On the other hand, in the case in which users submit to the network one single bid for the total amount of service desired, [4] shows that the efficiency loss can be arbitrarily large.

Kelly's model has also been investigated from the perspective of implementation theory [7, 9, 18]. These results present mechanisms which, at Nash equilibrium messages, achieve allocations that maximize the sum of the users' gross utility functions (their utility not counting what they pay). This is certainly a start on the implementation problem but it is not a complete answer. The problem lies in the role of

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[1] In the case in which agents behave strategically we need to specify an equilibrium concept used by the mechanism for generating allocations. In these cases the equilibrium concept was Nash. We discuss these concepts in Section III-A.2.

[2] An allocation's efficiency loss is measured by the ratio between the sum or the user's utility functions corresponding to that particular allocation and the sum of the user's utility functions corresponding to an optimal allocation.

the payments of the users to the network. In the realization approach, it is implicitly assumed that these payments are not important, they are simply a part of the computations necessary to provide information between the users and the network. In the implementation approaches of [7, 9, 18], it is assumed that the payments are important, at least to the users, since the users base their strategic calculations on net utility (gross utility minus the payments). The natural interpretation is that the users are actually paying money to the networks. If this is true and if one wants to maximize the welfare of the users, the appropriate concept of efficiency is based on the net utilities of the users. That is, the network should maximize the sum of the users' net utilities instead of the sum of their gross utilities. When a user's gross utility is his willingness to pay for an allocation, and his net utility is simply his gross utility minus his payment for those allocations, maximizing the sum of net utilities is equivalent to choosing an allocation that maximizes the sum of gross utilities and having a set of payments that add up to zero. The condition that payments add up to zero is known in the economics literature as the "balanced budget" condition. The mechanisms in [7, 9, 18] are not budget balanced as they overcharge the users for the allocations provided. So they are not efficient from the users' point of view.

Generically, there are no mechanisms that implement, in dominant strategies, the allocation that maximizes the sum of the users' net (and gross) utilities and satisfy budget balance. But there are mechanisms for public goods economies, now called Groves-Ledyard mechanisms [2], that do balance the budget and yield allocations that maximize the sum of the user's net utilities at Nash equilibrium. They do this with a variation on Vickery-Clarke-Groves (VCG) mechanisms. They ask consumers to communicate a one-parameter demand function for the public good.[3] The public good is chosen to equate demand and supply in the usual way for public goods and payments are the VCG payment plus a lump sum term for each consumer that balances the budget. Nash equilibria exist [3] and are Pareto-optimal.

Groves-Ledyard mechanisms may, however, leave a user worse off than if they did not use the system at all; that is, their payment may be higher than the gross utility of their allocation. In this case we say the mechanism violates "vountary participation". There are also mechanisms for economies, see for example [5], that do balance the budget, satisfy voluntary participation and maximize the sum of net utilities at Nash Equilibrium. It has been an open question whether such mechanisms exist for the rate allocation problem.

In this paper we present a pricing mechanism for the rate allocation problem that (i) generates efficient allocations, allocations that maximize both the sum of net utilities and the sum of gross utilities, (ii) is budget balanced, and (iii) satisfies voluntary participation.

Since multiple mechanisms may solve an informationally decentralized resource allocation problem, an important research topic is to classify these mechanisms in terms of their "communication" and "information processing" requirements. In this paper we show that, under the definition of informational efficiency presented in [16], the pricing mechanism developed in this paper is informationally efficient.

The rest of the paper is organized as follows: We formulate a network resource allocation problem in Section II. We briefly present the salient features and the desired properties of mechanisms from both the realization and implementation theory perspective in Section III. In Section IV we develop a pricing mechanism for the problem formulated in Section II which satisfies the properties presented in Section III. We investigate the informational efficiency of our pricing mechanism in Section V. We discuss other features of our mechanism in Section VI and we conclude in Section VII.

II. Problem Formulation

Consider a set of $\mathbf{N} = \{1, 2, \dots, N\}$ users requesting services from a network. Each user $i \in \mathbf{N}$ requests a single service from the network. User i's preference over the service rate received is summarized by a quasi linear utility function[4] of form $U_i(x_i, y_i) = u_i(x_i) + y_i$, with $x_i \in \mathbb{R}_+$. $U_i(x_i, y_i)$ is called user i's *net utility function*, and $u_i(x_i)$ is called user i's *gross utility function*. We assume that $u_i(x_i)$ is a differentiable and strictly concave function of x_i, with $u_i(0) = 0$.[5] Define $x \triangleq (x_1, x_2, \dots, x_N)$ to be the vector of demands requested by the N users. Denote a user and its service by the same index.

Denote by $\mathbf{L} \triangleq \{l_1, l_2, \dots, l_K\}$ the set of links in the network, with K representing the number of links in the network. For every $l \in \mathbf{L}$ denote by c_l the amount of resource/bandwith available on link l.

The problem's objective is to devise a network's rate allocation mechanism which maximizes the social welfare function described by the sum of the user utilities. Hence, the goal of the network is:

$$\max_{x,y} \sum_{i \in \mathbf{N}} U_i(x_i, y_i) \qquad \mathbf{P}$$

subject to:

$$x_i \geq 0, \quad \forall\ i \in \mathbf{N} \qquad \mathbf{P.a}$$

$$\sum_{i \in \mathbf{N}} x_i r_{i,l} \leq c_l, \quad \forall\ l \in L \qquad \mathbf{P.b}$$

$$\sum_{i \in \mathbf{N}} y_i \leq 0 \qquad \mathbf{P.c}$$

where

$$r_{i,l} \triangleq \begin{cases} 1, \text{ if user } i\text{'s request utilizes link } l \\ 0, \text{ otherwise} \end{cases}$$

Constraint **P.a** assures that each user receives a non-negative amount of service, **P.b** is the link capacity constraint, and **P.c** is the numeraire feasibility constraint (i.e. the amount of money

[3]For the public good problem these functions are of the form $x^i = \gamma(a^i - \frac{y}{I})$ where γ is fixed and known to everyone and I is the number of consumers. In the public good world, (Lindahl) market equilibrium requires $\sum x^i = 0$ or $\sum a^i = y$.

[4]A function $f(x, y)$ is called *quasi-linear* in y if it is of the form $f(x, y) = g(x) + y$. Parameter y is called the *numeraire* commodity, and in a market economy it generally has the interpretation of money.

[5]We assume that zero amount of service gives zero amount of utility.

in the system after the allocation can not exceed the amount before the allocation).

The main difficulty in the design of resource allocation policies for problems such as $\mathbf{P}$ comes from the informationally decentralized nature of the network problem. If information was to be centralized one could use mathematical programming techniques to determine a solution for problem $\mathbf{P}$. Since information is not centralized one must develop a message exchange process and an allocation function which maps "equilibrium messages" to solutions of problem $\mathbf{P}$. We will call the message exchange process along with the allocation function a resource allocation mechanism.

In the next section we present some of the components and desired properties of resource allocation mechanisms.

III. Mechanism Design

A. Mechanism Components

Formally,[6] resource allocation problems can be described by the following triple: environment, action space, and goal correspondence. We define the *environment* $\mathbf{E}$ of such problems to be the set of individual endowments, the technology, and preferences, taken together. More generally, the environment is defined as the set of circumstances that cannot be changed either by the designer of the mechanism or by the agents. The *action space* $\mathbf{A}$ of the problem is considered to be the set of all possible actions (e.g. resource allocations) conducted by the various agents. Finally, the *goal correspondence* π is the map from $\mathbf{E}$ to $\mathbf{A}$ which assigns for every $e \in \mathbf{E}$ the set of efficient actions in $\mathbf{A}$ (i.e. the set of "desired" solutions to the resource allocation problem).

In the context of the network Problem $\mathbf{P}$, the problem components are as follows:

i. environment $\mathbf{E}$ - each element of $\mathbf{E}$ represents an instance of Problem $\mathbf{P}$ (i.e. a network topology, a set of link capacities, and a set of user utility functions);

ii. action space $\mathbf{A}$ - the set of user service rates;

iii. goal correspondence π - a map from $\mathbf{E}$ to $\mathbf{A}$ that picks for each element in $\mathbf{E}$ an element of $\mathbf{A}$ which maximizes Problem $\mathbf{P}$.

The setup described above corresponds to the case in which one of the agents has enough information about the environment so as to determine the actions that would satisfy the goal correspondence (i.e. the information in the systems is centralized). Generally this is not the case. Usually, different agents have different information about the environment (i.e. we have an informationally decentralized system). For this reason it is necessary to devise a message exchange process among the various agents that eventually enables them to jointly take an action which corresponds to a solution of the centralized problem. We call such a process of communication, decisions and actions a *resource allocation mechanism*.

The function of resource allocation mechanisms is to guide the agents (economic or otherwise) to make decisions that determine the flow of resources. The design of these mechanisms has been approached from two perspectives: realization theory and implementation theory. We present each one of these perspectives separately.

1) Realization Theory: Formally, in realization theory, a resource allocation mechanism can be described by the following triple $(\mathcal{M}, \mu, h)$: a message space $\mathcal{M}$, an equilibrium message correspondence μ, and an outcome function h. The message space is the set of messages that may be exchanged by the agents. The equilibrium message correspondence describes the sets of messages that the agents "agree" upon given any particular environment. The outcome function describes the actions that are taken based on a particular set of "equilibrium" messages. The above formulation is depicted graphically in Figure 1 (cf. [14]).

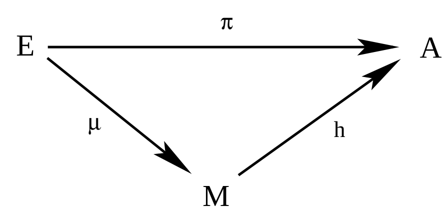

Fig. 1. Mechanism components in Realization Theory.

Realization theory is concerned with existence and design of mechanisms $(\mathcal{M}, \mu, h)$ such that the diagram in Figure 1 commutes. i.e.

$$h(\mu(e)) \subseteq \pi(e), \quad e \in E. \tag{III.1}$$

We call a mechanism satisfying equation (III.1) *goal realizing*. Within the context of resource allocation problems in communications networks multiple goal realizing mechanisms have been developed in the recent literature. The mechanisms which have been studied within the context of Problem $\mathbf{P}$ (and its variations) have been mainly pricing mechanisms. These mechanisms generally converge to an optimal feasible allocation by the means of a Tâtonnement process which can be described as follows:

1) To every user $i \in \mathbf{N}$ requesting service, the network computes and communicates a service price p_i.
2) Each user i computes the amount of rate x_i as to maximize its individual utility function. ($x_i \triangleq \operatorname{argmax}_x u_i(x) - p_i \times x$)
3) The network checks the excess demand on each one of its links, based on which he updates the service prices p_i.
4) The process repeats itself.

The mechanism components for the pricing mechanisms described above are:

i. message space $\mathcal{M}$ - $\mathcal{M} \triangleq \prod_{i=1}^{N} \mathcal{M}_i$, where $\mathcal{M}_i \triangleq \{(x_i, p_i)\}$ is the set of user's i (demand $\times$ price) pairs;

ii. outcome function h - projects the equilibrium (demand $\times$ price) pairs back to the demands;

iii. equilibrium message correspondence μ - determines, through the use of the Tâtonnement process, the equilibrium (demand $\times$ price) pairs.

[6] A more detailed presentation of the mechanism components and the design of resource allocation mechanisms in the context or communication networks can be found in [17].

One of the major weaknesses of the abovementioned pricing mechanisms comes from the fact that they do not take into account the strategic behavior of individual agents. The strategic behavior of an agent, within the context of a pricing mechanism, can be viewed as follows: User $i \in \mathbf{N}$ may realize that his individual service price p_i is a function of the his requested rate x_i. In this case, user i's best strategy is not to behave as a *price taker* (i.e. request a rate which solves $\text{argmax}_x\, u_i(x) - p_i \times x$), but rather request a lower rate in order to drive down the service price p_i. Through such strategies user i may be able to influence and significantly lower his service price p_i at the penalty of receiving a slightly lower service rate x_i.

Implementation theory takes account of the strategic behavior of individual agents. In the next section we present some of the features of resource allocation mechanisms from an implementation point of view.

2) Implementation Theory: The theory of implementation is concerned with the strategic behavior of allocation procedures, and generally studies implicit enforcing rules. Particularly, it is concerned with the design of "game forms" that implement, in some behavioral equilibrium, social choice rules.

Specifically, N-agent *game forms* are defined by $(\mathcal{M}, h)$, where $\mathcal{M} = \prod_{i=1}^{N} \mathcal{M}_i$, $\mathcal{M}_i$ is the message space of agent i, $i = 1, 2, \ldots, N$, and $h : \mathcal{M} \to \mathbf{A}$. Thus, for each profile $m := (m_1, m_2, \ldots, m_N)$ of messages, $h(m) \in \mathbf{A}$ represents the resulting outcome or allocation.

Define a *preference profile* to be a complete, binary and reflexive preordering $\mathcal{R}(e_i)$, that describes agent i's preferences over alternatives in $\mathbf{A}$ when i's environment is $e_i \in \mathbf{E}_i$.[7] A game form along with a specified preference profile for each user $i = 1, 2, \ldots, N$ induces a game.

A *solution concept* specifies the strategic behaviors of agents faced with a game form $(\mathcal{M}, h)$ given a preference profile $\mathcal{R}(e) := (\mathcal{R}(e_1), \mathcal{R}(e_2), \ldots, \mathcal{R}(e_N))$. Hence, a solution concept is a correspondence Λ that identifies a subset of $\mathcal{M}$ for any given specification $(\mathcal{M}, h, \mathcal{R}(e))$. We represent the components of a mechanism from the implementation theory point of view in Figure 2.

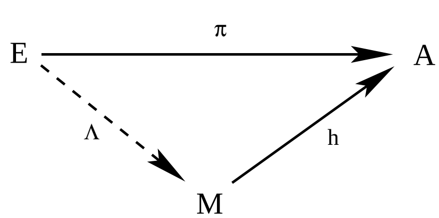

Fig. 2. Mechanism components in Implementation Theory.

We note that the main difference in the design of resource allocation mechanisms from realization and implementation theory point of view lies with the specification of the equilibrium message correspondence. While in realization theory the equilibrium message correspondence is a mechanism parameter which is prescribed by the mechanism designer, in implementation theory the equilibrium message correspondence is induced from the user's preference profile along with the mechanism's message space and outcome function.

We define

$$\mathfrak{O}_\Lambda := \{a \in \mathbf{A} | \exists m \in \Lambda(\mathcal{M}, h, \mathcal{R}(e)) \text{ s.t. } h(m) = a\} \quad \text{(III.2)}$$

to be the set of outcomes associated with the solution concept Λ. In order to illustrate equation (III.2) we present two examples of equilibrium criteria as solution concepts.

Example 3.1 (Dominant Strategies):

For any given $(\mathcal{M}, h, \mathcal{R}(e))$ a *dominant strategy* equilibrium is a message

$$m := (m_1, m_2, \ldots, m_N) \in \mathcal{M}$$

such that for all $i \in \mathbf{N}$, all $\overline{m}_i \in \mathcal{M}_i$, and all

$$m^*_{-i} := (m^*_1, \ldots, m^*_{i-1}, m^*_{i+1}, \ldots, m^*_N) \in \prod_{\substack{j \in \mathbf{N} \\ j \neq i}} \mathcal{M}_j$$

we have

$$h(m_i, m^*_{-i}) \mathcal{R}(e_i) h(\overline{m}_i, m^*_{-i}).^8 \quad \text{(III.3)}$$

Denote the messages satisfying (III.3) by $\mathbf{DS}(\mathcal{M}, h, \mathcal{R}(e))$. Then, the set of associated outcomes is

$$\mathfrak{O}_{\mathbf{DS}(\mathcal{M},h,\mathcal{R}(e))} := \{a | \exists m \in \mathbf{DS}(\mathcal{M}, h, \mathcal{R}(e)) \text{ s.t. } h(m) = a\} \quad \text{(III.4)}$$

Example 3.2 (Nash Equilibrium):

For any given $(\mathcal{M}, h, \mathcal{R}(e))$ a *pure Nash equilibrium* is a message

$$m := (m_1, m_2, \ldots, m_N) \in \mathcal{M}$$

such that

$$h(m) \mathcal{R}(e_i) h(\overline{m}_i, m_{-i}) \quad \text{(III.5)}$$

for all $i = 1, 2, \ldots, N$, and all $\overline{m}_i \in \mathcal{M}_i$, where

$$m_{-i} := (m_1, m_2, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n).$$

Denote the messages satisfying (III.5) by $\mathbf{NE}(\mathcal{M}, h, \mathcal{R}(e))$. Then, the set of associated outcomes is

$$\mathfrak{O}_{\mathbf{NE}(\mathcal{M},h,\mathcal{R}(e))} := \{a | \exists m \in \mathbf{NE}(\mathcal{M}, h, \mathcal{R}(e)) \text{ s.t. } h(m) = a\} \quad \text{(III.6)}$$

To define precisely how social choice correspondences are implicitly enforced via game forms in some behavioral equilibrium we need the following:

Definition 3.1: A social choice correspondence $\pi : \mathbf{E} \to \mathbf{A}$ is *implemented* by the game form $(\mathcal{M}, h)$ via the solution concept Λ if

$$\mathfrak{O}_\Lambda(\ (\mathcal{M}, h, \mathcal{R}(e))\ ) = \pi(e) \quad \text{(III.7)}$$

for all $e \in \mathbf{E}$.

Note that within realization theory the equivalent concept for a mechanism to implement a solution is for it to be goal realizing.

[7] Note that if user i has a utility function U_i over the set of resource allocations, then this utility function induces a preference profile $\mathcal{R}(e_i)$.

[8] $a\mathcal{R}(e_i)b$ denotes that under the preference relation induced by environment e_i, user i prefers allocation a over allocation b.

B. Desired Properties of Mechanisms

In this section we present some of the desired properties of the outcomes generated by resource allocation mechanisms. Particularly, we are going to only consider mechanisms which generate outcomes that are: (i) feasible; (ii) satisfy voluntary participation; and (iii) efficient. These are well established properties in the mathematical economics community.

1) Definitions:

Definition 3.2: We call an allocation vector $(x, y) \triangleq ((x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N))$ for Problem $\mathbf{P}$ *feasible* if it satisfies constraints **P.a**, **P.b** and **P.c**.

Definition 3.3: A feasible allocation (x, y) is said to satisfy *voluntary participation* if

$$U_i(x_i, y_i) \geq 0, \forall i \in \mathbf{N}.$$

The interpretation of feasible allocation is that it satisfies the physical constraints of the resource allocation problem. The interpretation of voluntary participation is that no user participating in the resource allocation process will become worse off at the end of the allocation process (i.e. each user will have a non-negative utility for participating in the allocation process).

We now present a few different efficiency criteria for allocation processes.

Definition 3.4: A feasible allocation

$$(x^*, y^*) \triangleq ((x_1^*, y_1^*), (x_2^*, y_2^*), \ldots, (x_N^*, y_N^*))$$

of problem $\mathbf{P}$ is said to be *efficient* (or utility maximizing) if for any other feasible allocations

$$(x, y) \triangleq ((x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N))$$

we have

$$\sum_{i \in \mathbf{N}} U_i(x_i^*, y_i^*) \geq \sum_{i \in \mathbf{N}} U_i(x_i, y_i).$$

Definition 3.5: A feasible allocation (x^*, y^*) of problem $\mathbf{P}$ is said to be *Pareto efficient* if there does not exist another feasible allocation (x, y) such that

$$U_i(x_i, y_i) \geq U_i(x_i^*, y_i^*) \quad \forall i \in \mathbf{N}$$

with the inequality being strict for some i.[9]

Definition 3.6: Feasible allocation (x^*, y^*) is called *output efficient* if for any other feasible allocation (x, y) we have:

$$\sum_{i \in \mathbf{N}} u_i(x_i^*) \geq \sum_{i \in \mathbf{N}} u_i(x_i).$$

We first note that in general, efficient allocations imply Pareto efficient allocations. One can easily prove that in the context of concave quasi-linear utility functions, Pareto efficient allocations are efficient.

In the next section we are going to investigate the relationship between efficient allocations and output efficient allocations.

[9]The definition of efficiency used in this paper is not standard. In some of the economic literature the definition of Pareto efficiency is used for efficiency.

2) Relationship between efficiency and output efficiency:

Before we present the relationship between efficient allocations and output efficient allocations we need the following definition.

Definition 3.7: An allocation

$$(x, y) \triangleq ((x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N))$$

is called *balanced* if constraint **P.c** is satisfied with equality.

The following lemma gives us the relationship between the efficiency and output efficiency.

Lemma 3.1: In the case of quasi-linear utility functions, an allocation is efficient if and only if it is output efficient and balanced.

Proof:

The proof of this lemma is in Appendix I. ■

By definition an output efficient allocation is not concerned with how the numeraire commodity is distributed as long as it satisfies constraint **P.c**. Particularly, given an efficient allocation, if one of the agents "burns" some of his numeraire commodity the resulting allocation ceases to be efficient but remains output efficient.

3) Implementation in Dominant Strategies vs. Nash Equilibria:

We note that by definition dominant strategies is a stronger equilibrium concept then Nash. It seems that given an informationally decentralized problem one should first attempt to develop a mechanism which implements it in dominant strategies before considering Nash implementation. It turns out, as we discuss below, that this is actually not the case.

In the case in which the agents (users) have quasi-linear utility functions, all the mechanisms which implement in dominant strategies resource allocation problems fall within the class of mechanisms developed by Vickrey-Clarke-Groves (VCG) [11]. For many applications these mechanisms are quite impractical for the following reasons:

1) **Informational grounds:** A major problem with the VCG mechanism, from a practical perspective, is the fact that each user must communicate to a central agent his entire profile. In the case of Problem $\mathbf{P}$, a user's profile is his utility function. Generally, a utility function could be rather complicated and may require an infinite number of parameters to fully describe it. Such a large amount of information could be impossible for the users to transmit and for the central agent to store.
2) **Complexity of information processing:** Assuming that a central agent has received all the users' profiles, determining the dominant strategy equilibrium is computationally extraneous.
   Particularly, in computing the allocation of each user $i \in \mathbf{N}$, the central agent must solve the problem multiple times (once when all users take part in the problem, and once when user i is excluded from the problem). If the problem contains a large number of users, the computation of such a large number of centralized solutions could be infeasible.
3) **Lack of efficiency:** The allocations generated by the VCG mechanism are output efficient, but not efficient. This comes from the fact that in order for the agents

to "agree" on the social optimal resource allocation, the budget balanced constraint will generally be violated. If the problem's objective is to determine an efficient outcome, implementation in dominant strategies is not possible.

For the above reasons we will not consider dominant strategies as a desirable equilibrium concept for implementing Problem $\mathbf{P}$. In the next section we develop a mechanism which implements the rate allocation problem $\mathbf{P}$ in Nash equilibria.

IV. A Pricing Mechanism for Unicast

In this section we are going to develop a mechanism which is efficient, satisfies voluntary participation and implements in Nash equilibria Problem $\mathbf{P}$. We assume that the number or agents interested in each commodity is larger than two. Since the number of users on each link of a network tends to be large, we believe that this assumption is not restrictive.

For illustrative purposes, in Section IV-A we are going to present the mechanism for the case of a single link network. We generalize this mechanism for the multi-link case in Section IV-B. In Section IV-C we provide some intuition behind the main ideas of our mechanism. This intuition could be used in the development of other mechanisms which implement rate allocation problem $\mathbf{P}$ in Nash equilibria.

A. One link model

Consider the case of a network consisting of a single link with capacity c. Assume that all the user services are being delivered via this link. Under these assumptions Problem $\mathbf{P}$ becomes:

$$\max_{x,y} \sum_{i\in\mathbf{N}} U_i(x_i, y_i) \qquad \mathbf{P}'$$

subject to:

$$x_i \geq 0, \quad \forall\, i \in \mathbf{N} \qquad \mathbf{P}'.\mathbf{a}$$

$$\sum_{i\in\mathbf{N}} x_i \leq c \qquad \mathbf{P}'.\mathbf{b}$$

$$\sum_{i\in\mathbf{N}} y_i \leq 0 \qquad \mathbf{P}'.\mathbf{c}$$

1) Pricing Mechanism: Assume that we endow each one of the N users with $\frac{c}{N}$ of the available bandwidth. The users then trade bandwidth among themselves in order to maximize their own individual revenue. Consider the following pricing mechanism:

Messages: Each user $i \in \mathbf{N}$ submits a message (x_i, p_i) to the network, where x_i represents the amount of bandwidth desired, and p_i represents the user's valuation of the "price" per unit of bandwidth.

To each user $i \in \mathbf{N}$, the network submits a message (p_{-i}, d_i, γ) where

$$p_{-i} \triangleq \sum_{\substack{j\in\mathbf{N}\\ j\neq i}} \frac{p_j}{N-1}$$

is the average of the other users' price per unit of bandwidth, γ is a positive constant,[10] and

$$d_i \triangleq \sum_{\substack{j\in\mathbf{N}\\ j\neq i}} x_j - c$$

is the excess demand when user's i demand is eliminated.

Allocations: Given an equilibrium set of messages

$$(x^*, p^*) \triangleq ((x_1^*, p_1^*), (x_2^*, p_2^*), \ldots, (x_N^*, p_N^*))^{11}$$

each user i is allocated the requested bandwidth x_i^*[12] and is taxed as follows:

$$\mathbf{t}_i(x^*, p^*) \triangleq (x_i^* - \frac{c}{N}) \times p_{-i}^* + \big[p_i^* - p_{-i}^*(1 + \frac{\sum_{j\in\mathbf{N}} x_j^* - c}{\gamma}) - \chi_+(x^*, c, \gamma)\big]^2 \qquad \text{(IV.1)}$$

where

$$\chi_+(x^*, c, \gamma) \triangleq \big(\max\{0, \frac{\sum_{j\in\mathbf{N}} x_j^* - c}{\gamma}\}\big)^2.$$

2) Implementation: The next result proves that even in the case in which agents behave strategically, the equilibrium allocations generated by the mechanism presented above are efficient.

Theorem 4.1: The mechanism presented in Section IV-A implements in Nash equilibria Problem $\mathbf{P}'$.

Proof: The proof of this theorem is in Appendix II. ■

In the following lemma we show that the Nash equilibrium allocations generated by the mechanism in this section satisfies voluntary participation property. Particularly, this result proves that each user prefers the Nash equilibria allocations over not participating in the allocation process.

Lemma 4.1: The mechanism presented in Section IV-A satisfies voluntary participation.

Proof: Fix the prices and demands of all users except user i. Then,

$$\max_{x_i, p_i} u_i(x_i) - \Big[(x_i - \frac{c}{N}) \times p_{-i} + \big[p_i - p_{-i}(1 + \frac{\sum_{j\in\mathbf{N}} x_j - c}{\gamma}) - \chi_+(x_i, c, \gamma)\big]^2\Big] \geq$$

$$\max_{p_i} \underbrace{u_i(0)}_{=0} - \Big[\underbrace{(-\frac{c}{N}) \times p_{-i}}_{\leq 0} + \underbrace{\big[p_i - p_{-i}(1 + \frac{\sum_{j\in\mathbf{N}, j\neq i} x_j - c}{\gamma}) - \chi_+(0, c, \gamma)\big]^2}_{=0}\Big]$$

$$\geq 0.$$

[10] We will discuss the interpretation of γ in Section VI.

[11] For any $i \in \mathbf{N}$, x_i and p_i are assumed to be non-negative. If any user announces a negative x_i or p_i, then it is set by the network to 0.

[12] Note that x_i^* can be less then $\frac{c}{N}$. In this case we can think that user i is selling $\frac{c}{N} - x_i^*$ amount of bandwidth to the other users.

Since the above set of inequalities holds for arbitrarily fixed prices and demands of all users except i, it implies that for any Nash allocation the voluntary participation condition holds. ■

B. Multilink model

Define

$$\mathbf{L}_i \triangleq \{l | r_{i,l} = 1 \text{ and } l \in \mathbf{L}\}$$
$$\mathbf{I}_l \triangleq \{j | r_{j,l} = 1, j \in \mathbf{N}\}$$

where $\mathbf{L}_i$ represents the set of links used by service i, and $\mathbf{I}_l$ the set of services utilizing link l. We assume that for every $l \in \mathbf{L}$, $|\mathbf{I}_l| > 2$ (i.e. there are three or more users utilizing each link).[13] We are going to discuss how we can relax this assumption to $|\mathbf{I}_l| \neq 2$ in Section VI.

We will now present a pricing mechanism which implements Problem $\mathbf{P}$ in Nash equilibria. This mechanism is a generalization of the mechanism presented in Section IV-A.

Assume that for each $l \in \mathbf{L}$ we endow $\frac{c_l}{|\mathbf{I}_l|}$ amount of bandwidth to each $j \in \mathbf{I}_l$. By following the pricing mechanism presented below, the users then trade bandwidth among themselves in order to generate services which maximize their own individual revenue.

Messages: Each user $i \in \mathbf{N}$ submits a message (x_i, p_i) to the network, where x_i represents the amount bandwidth desired, and $p_i \triangleq \{p_{i,l}\}_{l \in \mathbf{L}_i}$ represents the vector of user's valuation of the "price" per unit of bandwidth on the links used by service i.

To each user $i \in \mathbf{N}$, the network submits a message (p_{-i}, d_i, γ) with:

$$p_{-i} \triangleq \{p_{-i,l} | l \in \mathbf{L}_i\}$$
$$d_i \triangleq \{d_{i,l} | l \in \mathbf{L}_i\}$$
$$\gamma \triangleq \{\gamma_l | l \in \mathbf{L}_i\}$$

where

$$p_{-i,l} \triangleq \sum_{\substack{j \in \mathbf{I}_l \\ j \neq i}} \frac{x_j}{|\mathbf{I}_l| - 1}$$
$$d_{i,l} \triangleq \sum_{\substack{j \in \mathbf{I}_l \\ j \neq i}} x_j - c_l$$
$$\gamma_l \in \mathbb{R}_{++} \quad \forall l \in \mathbf{L}$$

$p_{-i,l}$ represents the average of the user's announced prices per unit of bandwidth on link l when user i is excluded. $d_{i,l}$ represents the excess demand on link l minus the demand of user i. We provide an interpretation of γ_l in section VI.

Allocations: Given an equilibrium set of messages

$$(x^*, p^*) \triangleq ((x_1^*, p_1^*), (x_2^*, p_2^*), \dots, (x_N^*, p_N^*))$$

each user i is allocated the requested bandwidth x_i^* and is taxed as follows:

$$\begin{aligned}\mathbf{t}_i(x^*, p^*) \triangleq & \sum_{l \in \mathbf{L}_i} (x_i^* - \frac{c_l}{|\mathbf{I}_l|}) \times p_{-i,l}^* \\ & + \sum_{l \in \mathbf{L}_i} \big[p_{i,l}^* - p_{-i,l}^* \big(1 + \frac{\sum_{j \in \mathbf{I}_l} x_j^* - c_l}{\gamma_l}\big) \\ & - \chi_+(x^*, \mathbf{I}_l, c_l, \gamma_l) \big]^2 \end{aligned} \quad \text{(IV.2)}$$

where

$$\chi_+(x^*, \mathbf{I}_l, c_l, \gamma_l) \triangleq \big(\max\{0, \frac{\sum_{j \in \mathbf{I}_l} x_j^* - c_l}{\gamma_l}\} \big)^2.$$

Theorem 4.2: The mechanism presented above satisfies voluntary participation and implements Problem $\mathbf{P}$ in Nash equilibria.

Proof: The proof of this theorem is very similar to the proofs of Theorem 4.1 along with Lemma 4.1, and is omitted. ■

C. Discussion

In Section IV-B we proposed a mechanism which implements the rate allocation Problem $\mathbf{P}$ in Nash equilibria. The key idea behind designing such a mechanism is the construction of a tax function which satisfies the following properties:

1) for each user $i \in \mathbf{N}$, i's price per unit of service is not dependent on his messages;
2) at equilibrium the capacity constraints are satisfied;
3) at equilibrium the allocations generated are efficient.

Before we describe how our mechanism satisfies the above properties, we note that the tax function (IV.2) can be decomposed into two terms:

$$\mathbf{t}_i(x, p) \triangleq \mathbf{I} + \mathbf{II} \quad \text{(IV.3)}$$

where

$$\mathbf{I} \triangleq \sum_{l \in \mathbf{L}_i} (x_i - \frac{c_l}{|\mathbf{I}_l|}) \times p_{-i,l}$$
$$\mathbf{II} \triangleq \sum_{l \in \mathbf{L}_i} \big[p_{i,l} - p_{-i,l} \frac{\gamma_l + \sum_{j \in \mathbf{I}_l} x_j - c_l}{\gamma_l} - \chi_+(x, \mathbf{I}_l, c_l, \gamma_l) \big]^2$$

The role of term $\mathbf{I}$ is to determine how much each individual has to pay/receive for the bandwidth bought/sold. The role of term $\mathbf{II}$ is to "penalize" the users if: (i) they do not update the prices in a manner which forces all the prices to converge to the market clearing prices; and (ii) the capacity constraints are not satisfied.

The following remarks are direct consequences of the proofs of Theorems 4.1 and 4.2.

Remark 4.1: When maximizing his utility function, user $i \in \mathbf{N}$ chooses a vector of prices p_i which makes term $\mathbf{II}$ of $\mathbf{t}_i(x, p)$ equal to 0.

Remark 4.2: At equilibrium, for all $l \in \mathbf{L}$ and $i, j \in \mathbf{I}_l$, we have (i) $p_{i,l} = p_{j,l}$ and (ii) $p_{i,l} = p_{-i,l}$.

We will not discuss how our mechanism satisfies each one of the above properties.

[13] This assumption is not very restrictive since in most networks the number of services being delivered via each link tends to be large.

Property 1: Note that from Remark 4.1, at equilibrium, for every user $i \in \mathbf{N}$ only term **I** plays a role in how much user i must pay for his service. Since term **I** is not dependent on p_i, user i's best strategy is to behave as a price taker when requesting the amount of desired service x_i.
Property 2: This follows directly from Remarks 4.1 and 4.2.
Property 3: The equilibrium allocations generated by this mechanism are equivalent to the Walrasian allocations [11]. Since the set of constraints are convex and since the utility functions are assumed to be quasi-linear and strictly concave, the Walrasian allocations are efficient.

We note that there are many ways to construct a tax function which satisfies the above properties. Particularly, for every $i \in \mathbf{N}$, term **II** of $\mathbf{t}_i(x,p)$ can be generalized by writing it as a sum of functions $f_{i,l}(x_i, p_{i,l}, p_{-i,l}, d_{i,l})$ where:

$$\underset{p_{i,l}}{\operatorname{argmin}} f_{i,l}(x_i, p_{i,l}, p_{-i,l}, d_{i,l}) \begin{cases} > p_{-i,l}, \text{ if } x_i + d_{i,l} > 0 \\ < p_{-i,l}, \text{ if } x_i + d_{i,l} < 0 \\ = p_{-i,l}, \text{ if } x_i + d_{i,l} = 0 \end{cases} \quad \text{(IV.4)}$$

V. Informational Efficiency

In general there may exist multiple mechanisms which achieve an optimal solution to an informational decentralized problem. An important research topic in mechanism design is the development of methods for comparing the various mechanisms with each other. One such method, proposed by mathematical economists [10, 12], is to classify the mechanisms based on the size of their message space.

Definition 5.1: Given two mechanisms[14] $(\mathcal{M}, \mu, h)$ and $(\mathcal{M}', \mu', h')$, one says that $(\mathcal{M}, \mu, h)$ is *informationally superior* to $(\mathcal{M}', \mu', h')$ if the dimension of the message space $\mathcal{M}$ (denoted by $|\mathcal{M}|$) is less then or equal to the dimension of the message space $\mathcal{M}'$.

While comparing mechanisms based on the size of their message space, one must impose certain "regularity" conditions on the class of mechanisms considered, which are: (i) *privacy preserving*, and (ii) the mechanisms have maps μ and h which are *locally spot threaded*.

Definition 5.2: We say that a mechanism is *privacy preserving* if all of its agents generate their messages based only on their own information about the environment.

Definition 5.3: A correspondence $F : E \twoheadrightarrow M$ is *locally spot threaded* if for every $e \in E$ there exists an open set $U_e \subseteq E$, and a continuous function $f : U_e \to M$ such that $f(e') \in F(e')$ for all $e' \in U_e$.

The interpretation of the privacy preserving is that we would like to consider mechanisms where each agent generates his own messages based only on the information available to himself. The reason for requiring that the mechanisms satisfy the locally spot threaded requirement is that we would like to disallow the consideration of mechanisms which exert certain "information smuggling" behavior. Such behavior can be described as follows: Given an agent needing to communicate an n dimensional message to another agent, the information may be encoded in a m dimensional message, where $m < n$, by using some sort of Peano space filling technique. Unfortunately, such encoding technique tend to be highly discontinuous. Within the context of mechanism design, if μ or h fail the locally spot threaded criteria, it implies that a small "error" in approximating the environment may generate arbitrarily large "errors" in equilibrium messages and outcomes.[15]

Definition 5.4: We say that a mechanism $(\mathcal{M}, \mu, h)$ is regular if: (i) it is privacy preserving; and (ii) the μ and h are spot-threaded.

Definition 5.5: We say that a mechanism $(\mathcal{M}, \mu, h)$ is *informationally efficient* if: (i) it is regular and goal realizing; and (ii) for any other mechanism $(\mathcal{M}', \mu', h')$ which is regular and goal realizing we have $|\mathcal{M}| \leq |M'|$.

There are a few results which study the informational efficiency of mechanisms for various classes of problems [10, 12, 13, 16]. In [12] the authors prove that from the perspective of realization theory, in the case in which N agents are interested in allocating one good, and all agent's messages are broadcasted to all other agents, a pricing mechanism with a message space of dimension N is informationally efficient.[16] In [13] the authors show that informationally efficient mechanisms which implement a problem will have a message space at least as large as the message space of an informationally efficient mechanisms which realizes the problem. In particular, in [13] the authors show that the informationally efficient mechanism which implements the problem considered in [12] has a dimension equal to $N+1$.[17] In [10] the authors consider the more general case where messages are not broadcast. In this scenario the authors consider that each message is heard by only one agent. Within the context of the problem studied in [12], the authors show that the dimension of informationally efficient mechanisms which allocate a good among N agents needs to have a dimension of $2N-2$.[18]

In [16] the authors investigate the informational efficiency of mechanisms from the perspective of Problem $\mathbf{P}$. In particular the authors show that:

1) due to the nature of the information structure of the communication network problem, users are assumed to communicate directly only with the network (i.e. a user can not broadcast messages to all other users). Due to this communication constraint, any goal realizing mechanism must have a message space of dimension at least $2N$.

[14] In this section we present the concept of informational efficiency from the realization theory point of view. The same definitions hold in the case of implementation theory when the equilibrium massage correspondence μ is substituted by the induced map Δ.

[15] For a detailed discussion on the importance of spot-threaded condition the reader is referred to [15–17].

[16] In this mechanism one designates an agent which broadcasts to all other agents a price. Each one of the $N-1$ non-designated agents broadcasts to all other agents a demand.

[17] In this mechanism one designates an agent which broadcasts to all other agents a price and a one dimension message which is used in penalizing the agents if they deviate from the social norm. Each one of the $N-1$ non-designated agents broadcasts to all other agents a demand.

[18] In this mechanism one designates an agent which communicates a price to each one of the $N-1$ non-designated agents. Each one of the $N-1$ non-designated agents communicates a demand to the designated agent.

2) in the context of Problem $\mathbf{P}$, pricing mechanisms are regular.
3) pricing mechanisms of dimension $2N$, such as Kelly's, which realize problem $\mathbf{P}$, are informationally efficient. [19]

An interesting research problem is to design a mechanism which implements Problem $\mathbf{P}$ and is informationally efficient. Note that from the above results, if a pricing mechanism implements Problem $\mathbf{P}$ and has a message space of dimension equal to $2N$, then it is informationally efficient.

In presenting such a mechanism consider the following scenario: The network, for conceptual reasons, is decomposed into N service providers and K links (see Fig. 3). We assume that each user is paired up with a single service provider. For simplicity we denote each service provider by the same index as the user with which it is paired. The messages exchange process works as follows: Each user $i \in \mathbf{N}$ communicates only with service provider i, and each service provider i communicates with links $\mathbf{L}_i$.

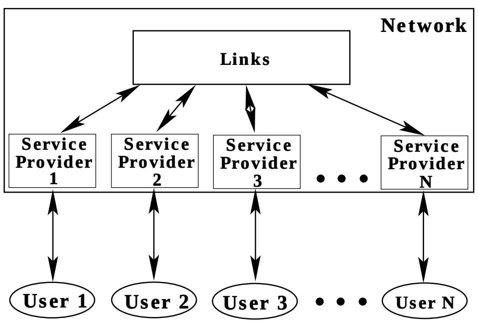

Fig. 3. An informationally efficient mechanism for Problem $\mathbf{P}$.

The message exchange process works as follows:
Links: Each link $l \in \mathbf{L}_i$ receives from service provider $i \in \mathbf{N}$ demand x_i and a link price $p_{i,l}$. It responds by transmitting to service provider i the triple $(p_{-i,l}, d_{i,l}, \gamma_l)$ defined in Section IV-B.
Service providers: Each service provider i receives from the users a demand x_i and from each link $l \in \mathbf{L}_i$ the triple $(p_{-i,l}, d_{i,l}, \gamma_l)$. The job of the service provider i is twofold:

1) it transmits to each link $l \in \mathbf{L}_i$ the user's demand x_i and a link price $p_{i,l}$ such that
$$p_{i,l} = p_{-i,l}\big(1 + \frac{\sum_{j\in \mathbf{I}_l} x_j - c_l}{\gamma_l}\big) + \chi_+(x, \mathbf{I}_l, c_l, \gamma_l),$$
2) it transmits to user i a price $\mathfrak{p}_i$, where
$$\mathfrak{p}_i \triangleq \sum_{l\in \mathbf{L}_i} p_{-i,l}.$$

Users: Each user $i \in \mathbf{N}$ receives a price $\mathfrak{p}_i$ from service provider i. Note that $\mathfrak{p}_i$ is generated from messages which are not a function of x_i. For this reason, it is in user's i best interest to behave as a price taker and request a demand x_i which solves:

$$x_i = \operatorname*{argmax}_x u_i(x) - \mathfrak{p}_i x. \tag{V.1}$$

Note that the equilibrium allocations generated by the above mechanisms coincide with the allocations generated by the mechanism presented in Section IV.

Under the new mechanism we note that the messages exchanged among the network and each user are a price and a demand (i.e. the message space of this mechanism has a dimension equal to $2N$). This implies that the above mechanism is informationally efficient. One may argue that the number of messages being communicated in the mechanism presented in this section and the one presented in Section IV is the same. We agree, but we note that most of the messages being exchanged in the mechanism presented in this section are among the components of a single agent - the network.

VI. Discussion

In this section we discuss some of the assumptions and issues which appear throughout the paper.
Form of the utility functions: In this paper we assumed that the user utility functions are quasi-linear, strictly concave and differentiable. These assumptions can be relaxed as follows:

Quasi-linearity - The mechanism presented in this paper charges the individual users a numeraire tax which encourages them to behave in a way which will induces an efficient outcome at the Nash equilibrium messages. We note that a similar tax function can be charged even in the case in which the user's utility functions are not quasi-linear. In such cases one can show that the mechanism's Nash equilibrium outcomes are Pareto efficient, but they need not be efficient.

Strict concavity - Strict concavity is a requirement that is generally needed when considering pricing mechanisms. One may be able to somewhat relax this assumption by requiring that all the efficient outcomes fall in a region where the objective function is strictly quasi-concave (i.e. the allocations lie in a region where the convex hull of the feasible region coincides with the feasible region).

Differentiability - We have assumed that the utility functions are differentiable. This assumption was used in the proof of Theorem 4.1 in writing the first order conditions. One can relax this requirement by expressing the first order conditions in terms of subgradients.
Interpretation of γ: In determining the Nash equilibrium allocations of Problem $\mathbf{P}$ one may use a Tâtonnement process, such as the one presented in Section III-A, where the users' tax function is as in Section IV-B. The interpretation of constant γ is the amount by which independent user link prices p_i are updated when the excess demand is non-zero. Note that for large γ's the amount by which the p_i's are updated is small, and for small γ's the amount is large. On the other hand, given too small of a γ, the process may start oscillating and never converge to an equilibrium allocation.

The optimal value for γ which maximizes the rate of convergence of the mechanism is based on a couple of factors: (i) the topology of the network; and (ii) the second derivative

[19] We note that in [7, 9, 18] the authors present mechanisms which seem to have message spaces of dimension N. In these papers the authors consider only the messages transmitted from the users to the network, and they do not account for the messages which need to be transmitted from the network to the users.

of the user utility functions. For the case in which users' utility functions have small second derivatives, small values of gamma may be acceptable.

Interpretation of p_i**:** We note that for every $l \in \mathbf{L}$, $p_{i,l}$'s do not have the direct interpretation of the link l's bandwidth price. Rather, they represent messages which correspond to user's i valuation of the bandwidth price. We note that only in equilibrium the $p_{i,l}$'s correspond to a link l's price per unit of bandwidth.

Interpretation of $\chi_+(x^*, \mathbf{I}_l, c_l, \gamma_l)$**:** The only reason for introducing $\chi_+(x^*, \mathbf{I}_l, c_l, \gamma_l)$ in the second term of the tax function is to eliminate the possible undesirable equilibria where: (i) for some $l \in \mathbf{L}$ all user's $p_{i,l}$'s are equal to 0; and (ii) the excess demand function on link l is positive.

Budget balanced property: In Lemma 3.1 we proved that a mechanism needs to be budget balanced in order for it to satisfy efficiency. We have also proved that at equilibrium messages our mechanism generates efficient outcomes. Unfortunately, our mechanism may not necessarily generate budget balanced outcomes from non-equilibrium messages. An interesting open problem is whether the mechanism can be generalized as to generate budget balanced allocations from non-equilibrium messages.

$|\mathbf{I}_l| \neq 1$ **requirement:** In Section IV-B, while presenting the pricing mechanism we have made the assumption that there is no link on which there exists exactly one service utilizing that link. We note that we can relax this assumption by altering the mechanism as follows: (i) each user $i \in \mathbf{N}$ receives a rate which is equal to the minimum among x_i and the capacities on the links on which service i is the only service; and (ii) user i is charged a zero price on the links on which service i is the only service.

Covering the network cost: In our model we have assumed that the network is formed by a set of links, each having a finite amount of bandwidth. Our objective has been to find the amount of bandwidth each user should receive in order to maximize the sum of the user's utilities. In our formulation we did not take into account the cost incurred by the network for providing the bandwidth allocation. In the case in which the cost of bandwidth on each link is linear (i.e. for each $l \in \mathbf{L}$ there exists a fixed price per unit of bandwidth q_l) one can easily integrate this cost inside the tax function by charging each user $i \in \mathbf{N}$ an extra tax $\sum_{l \in \mathbf{L}_i} q_l \times x_i$.

VII. Conclusion

In this paper we presented a pricing mechanism for an informationally decentralized network rate allocation problem. The mechanism developed satisfies the following properties: (i) it is efficient; (ii) it satisfies voluntary participation; and (iii) it implements the rate allocation problem in Nash equilibria. We provide some of the intuition behind this mechanism and we give some insight on the design of other mechanisms which satisfy the abovementioned properties.

We also present a measure for the informational efficiency of a mechanism and we describe a particular network instance under which our pricing mechanism is informationally efficient.

Appendices

Appendix I
Proof of Lemma 3.1

($:\Leftarrow$) Let

$$(x^*, y^*) \triangleq ((x_1^*, y_1^*), (x_2^*, y_2^*), \ldots, (x_N^*, y_N^*))$$

be an output efficient and balanced allocation. Then for any feasible allocation

$$(x, y) \triangleq ((x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N))$$

we have

$$\begin{aligned} \sum_{i \in \mathbf{N}} U_i(x_i^*, y_i^*) &= \underbrace{\sum_{i \in \mathbf{N}} u_i(x_i^*)}_{I} + \underbrace{\sum_{i \in \mathbf{N}} y_i^*}_{II} \\ &\geq \underbrace{\sum_{i \in \mathbf{N}} u_i(x_i)}_{III} + \underbrace{\sum_{i \in \mathbf{N}} y_i}_{IV} = \sum_{i \in \mathbf{N}} U_i(x_i, y_i) \end{aligned}$$

where $I \geq III$ since (x^*, y^*) is output efficient and $II \geq IV$ since (x^*, y^*) is balanced. This proves that (x^*, y^*) is efficient.

($:\Rightarrow$) We first prove that an efficient allocation has to be balanced. Assume by contradiction that there exists an efficient allocation for Problem $\mathbf{P}$

$$(x^*, y^*) \triangleq ((x_1^*, y_1^*), (x_2^*, y_2^*), \ldots, (x_N^*, y_N^*))$$

which is not balanced. Then

$$\sum_{i \in \mathbf{N}} y_i^* = -c < 0.$$

Consider allocation

$$\begin{aligned} (\tilde{x}, \tilde{y}) &\triangleq ((\tilde{x}_1, \tilde{y}_1), (\tilde{x}_2, \tilde{y}_2), \ldots, (\tilde{x}_N, \tilde{y}_N)) \\ &= ((x_1^*, y_1^* + c), (x_2^*, y_2^*), \ldots, (x_N^*, y_N^*)). \end{aligned}$$

Note that this allocation satisfies constraints **P.a** - **P.c** so it is feasible. Then

$$\begin{aligned} \sum_{i \in \mathbf{N}} U_i(x_i^*, y_i^*) &= \sum_{i \in \mathbf{N}} \Big(u_i(x_i^*) + y_i^* \Big) \\ &= u_1(x_1^*) + y_1^* + \sum_{2 \leq i \leq N} \Big(u_i(x_i^*) + y_i^* \Big) \\ &< u_1(x_1^*) + y_1^* + c + \sum_{2 \leq i \leq N} \Big(u_i(x_i^*) + y_i^* \Big) \\ &= \sum_{i \in \mathbf{N}} U_i(\tilde{x}_i, \tilde{y}_i) \end{aligned} \tag{I.1}$$

This contradicts the assumption that (x, y) is efficient.

We now prove that an efficient allocation has to be output efficient. Let

$$(x^*, y^*) \triangleq ((x_1^*, y_1^*), (x_2^*, y_2^*), \ldots, (x_N^*, y_N^*))$$

be an efficient allocation. Consider any other feasible allocation

$$(x, y) \triangleq ((x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N)).$$

We need to show that

$$\sum_{i\in\mathbf{N}} u_i(x_i^*) \geq \sum_{i\in\mathbf{N}} u_i(x_i)$$

.

Note that if (x, y) is feasible so is (x, y^*). Then by efficiency of (x^*, y^*), along with the fact that efficient allocations are balanced, we have:

$$\sum_{i\in\mathbf{N}} u_i(x_i^*) = \sum_{i\in\mathbf{N}} \Big(u_i(x_i^*)+y_i^*\Big) \geq \sum_{i\in\mathbf{N}} \Big(u_i(x_i)+y_i^*\Big) = \sum_{i\in\mathbf{N}} u_i(x_i)$$

which proves our result.

Appendix II
Proof of Theorem 4.1

In proving this theorem we proceed as follows. First we present the necessary and sufficient conditions for the efficient allocations of Problem $\mathbf{P}'$. Then we show that the Nash equilibria of mechanism presented in Section IV-A satisfy the efficiency conditions.

In order to determine an optimal solution of Problem $\mathbf{P}'$ we first write the Lagrangian function:

$$\Lambda(x, y, \lambda, \varphi) \triangleq \sum_{i\in\mathbf{N}} U_i(x_i, y_i) - \lambda(\sum_{i\in\mathbf{N}} x_i - c) - \varphi \sum_{i\in\mathbf{N}} y_i \tag{II.1}$$

At an optimal allocation $(x^*, y^*) \triangleq \{(x_i^*, y_i^*)\}_{i\in\mathbf{N}}$ the necessary and sufficient Karush-Kuhn-Tuker (KKT) conditions for optimality [1] are:

$$\frac{\partial}{\partial x_i}\Lambda(x^*, y^*, \lambda, \varphi) = \frac{\partial}{\partial x_i} u_i(x_i^*) - \lambda = 0, \forall i \in \mathbf{N} \tag{II.2}$$

$$\frac{\partial}{\partial y_i}\Lambda(x^*, y^*, \lambda, \varphi) = 1 - \varphi = 0, \forall i \in \mathbf{N} \tag{II.3}$$

$$\lambda(\sum_{i\in\mathbf{N}} x_i^* - c) = 0 \tag{II.4}$$

$$\varphi \sum_{i\in\mathbf{N}} y_i^* = 0 \tag{II.5}$$

where λ and φ are the Lagrange multipliers for the capacity and budget constraints, respectively.

Substituting equation (II.3) into equation (II.5), the KKT conditions can be reduced to:

$$\frac{\partial}{\partial x_i} u_i(x_i^*) - \lambda = 0, \forall i \in \mathbf{N} \tag{II.6}$$

$$\lambda(\sum_{i\in\mathbf{N}} x_i^* - c) = 0 \tag{II.7}$$

$$\sum_{i\in\mathbf{N}} y_i^* = 0 \tag{II.8}$$

where (II.6) states that at optimality the marginal utility equals to the Lagrange multiplier λ, (II.7) is the capacity constraint, and (II.8) states that the allocation must be budget balanced.

We are now going to investigate the Nash allocations generated by the mechanism presented in Section IV-A. In order to show that the mechanism implements in Nash equilibria Problem $\mathbf{P}'$ we need to show that the Nash allocations satisfy equations (II.6) - (II.8).

Note that for any $i \in \mathbf{N}$, given a fixed p_{-i}, $(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{|\mathbf{N}|})$, c and γ, user i picks x_i and p_i as to maximize his individual utility function:

$$\max_{x_i, p_i} u_i(x_i) - \Big[(x_i - \frac{c}{N}) \times p_{-i} + \big[p_i - p_{-i}(1 + \frac{\sum_{j\in\mathbf{N}} x_j - c}{\gamma}) - \chi_+(x, c, \gamma)\big]^2\Big] \tag{II.9}$$

At a Nash equilibrium message, equation (II.9) is maximized for each $i \in \mathbf{N}$. Assume that $(\tilde{x}, \tilde{p}) \triangleq \{(\tilde{x}_i, \tilde{p}_i)\}_{i\in\mathbf{N}}$ is a Nash equilibrium message. By the first order conditions for each $i \in \mathbf{N}$ we have that:

$$\frac{\partial}{\partial x_i} u_i(\tilde{x}_i) - \Big[\tilde{p}_{-i} + 2\Big(\frac{\tilde{p}_{-i}}{\gamma} + \frac{\partial}{\partial x_i}\chi_+(\tilde{x}, c, \gamma)\Big) \times \big[\tilde{p}_i - \tilde{p}_{-i}(1 + \frac{\sum_{j\in\mathbf{N}} \tilde{x}_j - c}{\gamma}) - \chi_+(\tilde{x}, c, \gamma)\big]\Big] = 0 \tag{II.10}$$

$$\tilde{p}_i - \tilde{p}_{-i}(1 + \frac{\sum_{j\in\mathbf{N}} \tilde{x}_j - c}{\gamma}) - \chi_+(\tilde{x}, c, \gamma) = 0. \tag{II.11}$$

where

$$\tilde{p}_{-i} \triangleq \sum_{\substack{j\in\mathbf{N}\\ j\neq i}} \frac{1}{N-1}\tilde{p}_j.$$

Substituting (II.11) into (II.10), the first order conditions become:

$$\frac{\partial}{\partial x_i} u_i(\tilde{x}_i) = \tilde{p}_{-i} \tag{II.12}$$

$$\tilde{p}_i = \tilde{p}_{-i}(1 + \frac{\sum_{j\in\mathbf{N}} \tilde{x}_j - c}{\gamma}) + \chi_+(\tilde{x}, c, \gamma) \tag{II.13}$$

Since (II.13) has to be satisfied for all $i \in \mathbf{N}$, by summing over all i we have:

$$\sum_{i\in\mathbf{N}} \tilde{p}_i = \sum_{i\in\mathbf{N}} \big[\tilde{p}_{-i}(1 + \frac{\sum_{j\in\mathbf{N}} \tilde{x}_j - c}{\gamma}) + \chi_+(\tilde{x}, c, \gamma)\big] \tag{II.14}$$

$$= \sum_{i\in\mathbf{N}} \sum_{\substack{j\in\mathbf{N}\\ j\neq i}} \frac{\tilde{p}_j}{N-1}(1 + \frac{\sum_{j\in\mathbf{N}} \tilde{x}_j - c}{\gamma}) + N \times \chi_+(\tilde{x}, c, \gamma) \tag{II.15}$$

$$= \sum_{i\in\mathbf{N}} \tilde{p}_i(1 + \frac{\sum_{j\in\mathbf{N}} \tilde{x}_j - c}{\gamma}) + N \times \chi_+(\tilde{x}, c, \gamma) \tag{II.16}$$

where (II.14) follows by (II.13), (II.15) by the definition of p_{-i}, and (II.16) by simplification.

Equations (II.14)-(II.16) imply that $\sum_{i \in \mathbf{N}} \tilde{x}_i = c$ and $\chi_+(\tilde{x}, c, \gamma) = 0$. This, along with equation (II.13) and the definition of $\tilde{p}_{-i}$, implies that at Nash equilibrium

$$\tilde{p}_i = \tilde{p}_{-i} \quad \forall i \in \mathbf{N} \tag{II.17}$$

$$\tilde{p}_i = \tilde{p}_j = \mathrm{p} \quad \forall i, j \in \mathbf{N} \tag{II.18}$$

Substituting (II.17) and (II.18) in (II.12) and (II.13) we have that:

$$\frac{\partial}{\partial x_i} u_i(\tilde{x}_i) = \mathrm{p} \tag{II.19}$$

$$\mathrm{p}(\sum_{j \in \mathbf{N}} \tilde{x}_j - c) = 0 \tag{II.20}$$

By letting $\mathrm{p} = \lambda$, equations (II.6) and (II.7) of the KKT first order conditions are satisfied. We now only have to show that the Nash equilibrium messages generates allocation which satisfy (II.8) (i.e. are budget balanced).

$$\begin{aligned}
\sum_{i \in \mathbf{N}} y_i &= \sum_{i \in \mathbf{N}} \mathbf{t}(\tilde{x}, \tilde{p}) \\
&= \sum_{i \in \mathbf{N}} (\tilde{x}_i - \frac{c}{N}) \times \tilde{p}_{-i} \\
&+ \sum_{i \in \mathbf{N}} \big[\tilde{p}_i - \tilde{p}_{-i}(1 + \frac{\sum_{j \in \mathbf{N}} \tilde{x}_j - c}{\gamma}) - \chi_+(\tilde{x}, c, \gamma)\big]^2 && \text{(II.21)} \\
&= \sum_{i \in \mathbf{N}} (\tilde{x}_i - \frac{c}{N}) \times \mathrm{p} + \sum_{i \in \mathbf{N}} \big[\mathrm{p} \times \frac{\sum_{j \in \mathbf{N}} \tilde{x}_j - c}{\gamma}\big]^2 && \text{(II.22)} \\
&= 0 && \text{(II.23)}
\end{aligned}$$

where (II.21) follows from equation (IV.1), equation (II.22) follows from equation (II.18), and equation (II.23) follows from equation (II.20).

This establishes that the Nash equilibrium allocations are balanced, which proves that the mechanism presented in Section IV-A implements in Nash equilibria Problem $\mathbf{P}'$.

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