From Nakamura Number to Stability Index

(Incomplete and preleminary)

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Abstract An interactive form is an abstract model of interaction that generalizes Simple Games and Effectivity Functions. The core, as a solution, as well as the correlated notion of stability can be extented to Interactive Forms. Necessary and sufficient conditions for stability can also be established in this setting, expressed by the absence of cycles. A Stability Index that plays a role similar to that of the Nakamura Number can be defined. This Index measures, loosely speaking, the strategic complexity of a an interactive form. If it is small than it is easy to find in the interacting society a preference profile that prevents the emergence of a solution. To any Strategic Game Form one can associate a special interactive form in such a way that given an equilibrium concept (Nash , strong Nash or others) and a preference profile, solutions of the interactive form are precisely the equilibrim outcomes of the game. As a consequence we have necessary and sufficient conditions for the solvability of the Game Forms. The model allows also a localization of the index in case of instability.

Keywords: Interactive Form, Stability Index, Nash Equilibrium, Strong Equilibrium, Solvability, Consistency, Simple Game, Effectivity Function, Acyclicity.

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1 Introduction

Game Theory provides a conceptual framework to the study of interaction between rational agents. Two models have been historically distinguished that differ by the primitives through which the interaction is described. In the strategic model, every agent is represented by an exhaustive set of available actions; the interaction is described as the mechanism that transforms the combined action choice into some outcome. In the so-called cooperative or coalitionnal model, the primitives are embedded in the description of the power of aggregates of agents ; the interaction is the simultaneous exercice of this power. In this paper we present a general model of interaction that goes beyond the strategy based / power based distinction, in that it allows in the same terms, for a representation of main coalitional concepts like simple games and effectivity functions, traditionnally presented as power-based models, and the essential features of strategic-based models where the outcome arises as the solution of some one-shot equilibrium concept.

Let N be a set of players (also called agents, individuals) and let A be a set of social states (also called alternatives). A coalition is a subset of players acting cooperatively. We shall illustrate informally our arguments by an example in politics by taking the case of a government formation in some State. Players are the atomic entities that are endowed with autonomous activity. In our example players are parties. In game theory a coalition is an aggregate of players that acts cooperatively. In this example a coalition is thus an entity formed of parties that take coordinated decisions. A government (to be formed) is any element of A. In a simple game power is described by the set $\mathcal{W}$ of winning coalitions. In this example a winning coalition is one that has the power to implement any government if it decides to do so. If $a \in A$ is a government, any coalition in $\mathcal{W}$ can thus oppose it by proposing any other goverment b and a loosing coalition can upset no government at all. Whether a winning coalition will object and act in consequence depends on its actual preferences. If no objection is formulated the government is adopted. In technical terms the outcome is in the core of the simple game given the preferences. Though simple games can fairly model some decision mechanisms, like weighted majority voting in some institutions, it is too simple to describe the political issues underlying a government formation. This is because a coalition is either absolutely powerfull or totally powerless. An effectivity function E allows a more general distribution of power among the coalitions. In our example, if $B \in E(S)$ where B is a subset of possible goverments and S is a coalition, then the latter can upset a by threatening to form some other goverment in B but without being able to precise which one. Why is it so ? - Because there are many ways to form the government a and the reaction of S must adapt to this fact. However here too, the power of a coalition does not depend on the current state a. This is a basic restriction in the model since we think that in such political interactions, a coalition may do some moves if the established goverment is a and some other moves if it were b. The idea is then to allow for effectivity power depending on the state a. This is the local effectivity function. Furthermore, when a government is formed the same outcome can be implemented in various ways as we said. Indeed different majorities or confederation of coalitions can implement the same outcome a, so that we have to define the joint power of coalitions in different majority confederations. If (S_1, S_2, S_3) are concerned by goverment formation, and if a can be formed either by (S_1, S_2) or (S_1, S_3), then either S_2 can upset it by proposing B_2 or S_3 can upset it by proposind B_3. Taking into account this idea will lead us to the concept of interactive form. In an interactive form we take into account the dependence of the interaction on the actual social state and the joint character of the reactive power of any confederation that contributes to it.

Given a preference profile for players, states that no confederation of coalitions have any interest to upset is called a settlement at that profile. The interactive form is said to be stable if it admits at least one settlement at any preference profile. Now assume that the interactive form is not stable as it is indeed so often the case in political life. Our idea is to define a measure of instability and for that purpose we introduce the stability index. The latter is a number that may be set to infinity in case of stability and that measures the difficulty to exclude settlements in the society. If this number is low, for instance two, then a simple split in the society with strong power on each side can lead to a stalemate at polarized preferences. If the index is high then unless some power to maneuver sophisticatedly is held by the agents at some intricate preference profile, a settlement can be reached

An interactive form can be associated to any strategic game form together with a given equilibrium concept. The idea of interactive form is similar to that of effectivity structures introduced by Abdou and Keiding [6]. The difference between the two models is that the present one allows for the representation of various equilibria concepts within the same interactive form, whereas the other is specific to one equilibrium concept. Since only the interactive form associated to a game form and an equilibrium concept is relevant for stability , a comparision between game forms that have the same players and the same alternative set is possible. An interactive form can thus be viewed as an intrinsic representation of power without a direct reference to strategies or an equilibrium concept.

The stability Index introduced in this paper plays a similar role as the Nakamura Number for simple games [12]. The difference is that the Nakamura number is defined on the winning coalition structure whereas the stability index depends on the whole interactive form.

2 Basic notations and definitions

2.1 Notations

Throughout this paper we shall consider a finite set N the elements of which are called players or agents and a finite set A the elements of which are called alternatives or social state. We make use of the following notational conventions in the sequel: For any set D, we denote by $\mathcal{P}(D)$ the set of all subsets of D and by $\mathcal{P}_0(D) = \mathcal{P}(D)\backslash\{\emptyset\}$ the set of all non-empty subsets of D. Elements of $\mathcal{P}_0(N)$ are called *coalitions*. $N\backslash S$ is denoted S^c. Similarly if $B \in \mathcal{P}(A)$, $A\backslash B$ is denoted B^c. $L(A)$ will denote the set of all linear orders on A (that is all binary relations on A which are complete, transitive, and antisymmetric). If $R \in L(A)$, and $a, b \in A, a \neq b$, $a\ R\ b$ means that a is better than b in the linear order R. A preference profile (over A) is a map from N to $L(A)$, so that a preference profile is an element of $L(A)^N$. For every preference profile $R_N \in L(A)^N$ and $S \in \mathcal{P}_0(N)$ we put

$$P(a, S, R_N) = \{b \in A \mid, b \neq a, b\ R^i\ a\ \forall i \in S\}$$

(so that $P(a, S, R_N)$ consists of all the outcomes considered to be better than a by all members of the coalition S), and $P^c(a, S, R_N) = A\backslash P(a, S, R_N)$.

2.2 Simple games and the Nakamura number

A nonempty subset $\mathcal{W}$ of $\mathcal{P}_0(N)$ is called a simple game if $\emptyset \notin \mathcal{W}$. Given any set A, $\mathcal{W}$ interacts on A as follows. Any coalition $S \in \mathcal{W}$ (a winning coalition) can react to any current state by imposing any other one. With this interpretation, the following definition is in order. Let $R_N \in L(A)^N$. An alternative $a \in A$ is *dominated* at R_N if there exists $S \in \mathcal{W}$ such that $P(a, S, R_N) \neq \emptyset$. The *core* of $(\mathcal{W}, A)$ at R_N is the set of undominated alternatives. It is denoted $C(\mathcal{W}, A, R_N)$. $\mathcal{W}$ is *stable* on A if $C(\mathcal{W}, A, R_N) \neq \emptyset$ for all $R_N \in L(A)^N$.

Definition 2.1 A family $(S_1, \ldots, S_r)$ where $S_k \in \mathcal{W}$ $(k = 1 \ldots, r)$ is called a *cycle* in $\mathcal{W}$ if $\cap_{k=1}^{k=r} S_k = \emptyset$.

The *length* of the cycle is the number r. We shall call it an r-cycle.

The *Nakamura Number* of $\mathcal{W}$ denoted $\nu(\mathcal{W})$ is defined as the minimum length of a cycle in $\mathcal{W}$. If $\mathcal{W}$ has no cycle we set $\nu(\mathcal{W}) = +\infty$

The following result was proved in Nakamura [12]

Theorem 2.2 *$\mathcal{W}$ is stable on A if and only if* $|A| < \nu(\mathcal{W})$

In view of next sections it is interesting to define the *stability index* for any couple $(\mathcal{W}, A)$ as follows:

$$\sigma_{(\mathcal{W},A)} = \nu(\mathcal{W}) \text{ if } \nu(\mathcal{W}) \leq |A| \tag{1}$$
$$= +\infty \text{ if } \nu(\mathcal{W}) > |A| \tag{2}$$

2.3 Effectivity Functions

In a simple game only coalitions in $\mathcal{W}$ have the power to oppose an alternative. Moreover this power is very sharp and unevenly distributed since on one hand a winning coalition can reach any alternative and on the other hand a loosing coalition (that is not in $\mathcal{W}$) has no power at all. In order to describe an interaction where the repartition of power among coalitions is more general, one can consider effectivity functions. In [11] this notion has been introduced in relation with stong implementation of social choice correspondences, in [1] an equivalent notion was defined as a generalization of simple games and veto functions. Here we provide a definition.

Definition 2.3 An *Effectivity function* on (N, A) is a mapping $E : \mathcal{P}(N) \rightarrow \mathcal{P}(\mathcal{P}_0(A)$ such that :

(i) $E(\emptyset) = \{A\}$

(ii) $S \in \mathcal{P}_0(N) \Rightarrow A \in E(S)$

(iii) $B \in E(S), B \subset B' \Rightarrow B' \in E(S)$

Let $R_N \in L(A)^N$. An alternative $a \in A$ is *dominated* at R_N if there exists $S \in \mathcal{P}_0(N)$ such that $P(a, S, R_N) \in E(S)$. The *core* of E at R_N is the set of undominated alternatives. It is denoted $C(E, R_N)$. E is *stable* if $C(E, R_N) \neq \emptyset$ for all $R_N \in L(A)^N$.

The action of a simple game $\mathcal{W}$ on a set A can be represented canonically as an effectivity function. This is done as follows: If $S \in \mathcal{W}$ let $E(S) = P_0(A)$, if $S \notin W$ let $E(S) = \{A\}$. Clearly the core of E and the core of $\mathcal{W}$ coincide for every preference profile. Conversly, given an effectivity function E, one can define the simple game induced by E, namely $\mathcal{W}$ is the set of coalitions S such that $E(S) = \mathcal{P}_0(A)$.

Definition 2.4 A *cycle* in E is an r-tuple $((U_1, B_1, S_1), \ldots, (U_r, B_r, S_r))$ where $r \geq 1$ $U_k \in P_0(A)$, $S_k \in P_0(N)$, $B_k \in E(S_k)$ $(k = 1, \ldots, r)$ and such that :

(i) $\cup_{k=1}^r U_k = A$

(ii) For any $\emptyset \neq J \subset \{1, \ldots, r\}$ such that $\cap_{k \in J} S_k \neq \emptyset$ there exists $k \in J$ such that for all $l \in J$ $U_k \cap B_l = \emptyset$

The *length* of the cycle is the natural number r. We call it then an r-cycle.

A cycle is *strict* if $(U_1, \ldots, U_r)$ is a partition of A. E is *acyclic* if E has no cycle.

We have the following result which was first proved by Keiding [7], see also [5]:

Theorem 2.5 *E is stable if and only if E is acyclic.*

Definition 2.6 The *stability index* of E, denoted $\sigma(E)$, is the minimal length of a cycle in E. This index is set to $+\infty$ if E is acyclic.

Since any cycle gives rise to a shorter strict cycle, it is clear that either $2 \leq \sigma(E) \leq |A|$ or $\sigma(E) = +\infty$.

An effectivity function E is *maximal* if $\forall S \in \mathcal{P}(N), \forall B \in \mathcal{P}(A)$,

$$B \notin E(S) \Longrightarrow B^c \in E(S^c) \tag{3}$$

E is *regular* if

$$S_1 \in \mathcal{P}(N), S_2 \in \mathcal{P}(N),\ B_1 \in \mathcal{P}(A),\ B_2 \in \mathcal{P}(A) \Rightarrow B_1 \cap B_2 \neq \emptyset \tag{4}$$

E is *superadditive* if

$$S_1 \cap S_2 = \emptyset,\ B_1 \in E(S_1),\ B_2 \in E(S_2) \Rightarrow B_1 \cap B_2 \in E(S_1 \cup (S_2) \tag{5}$$

E is *subadditive* if

$$B_1 \cap B_2 = \emptyset,\ B_1 \in E(S_1),\ B_2 \in E(S_2) \Rightarrow B_1 \cup B_2 \in E(S_1 \cap S_2) \tag{6}$$

The following result can be deduced from Abdou [1] and Peleg [13] (Theorem 6.A.9).

Theorem 2.7 *Let E be maximal. Then E is stable if and only if E is superadditive and subadditive.*

This provides a localization of the stability index for some effectivity functions:

Theorem 2.8 $\sigma(E) = 2$ *if and only if E is not regular*

For maximal effectivity functions we have:

Theorem 2.9 *Assume that E is maximal. E is not stable if and only if $\sigma(E) \leq 3$. If moreover E is regular then E is not stable if and only if $\sigma(E) = 3$*

2.4 Local effectivity functions

The distribution of power described in an effectivity function is global. A coalition is effective to some subsets of alternatives or not. In the study of game form solvability it is useful to introduce the idea that the power of a coalition depends on the starting point or the actual position. This is the reason why it was first introduced in [2]. But in any case, when modeling social interaction, it is highly realistic that acting (or reacting) power of the agents depends on the initial or default situation. This is why we introduce the following:

Definition 2.10 *A local effectivity function is a family $(E[a], a \in A)$ where for any $a \in A$ $E[a] : \mathcal{P}(N) \to P(P_0(A)$ the following conditions are satisfied:*
(i) $B \in E[a](\emptyset) \Leftrightarrow a \in B$
(ii) $S \in \mathcal{P}_0(N), a \in B \Rightarrow B \in E[a](S)$
(iii) $B \in E[a](S), B \subset B' \Rightarrow B' \in E(S)$

Let $R_N \in L(A)^N$. An alternative $a \in A$ is *dominated* at R_N if there exists $S \in \mathcal{P}_0(N)$ such that $P(a, S, R_N) \in E[a](S)$. The *core* of $E[\cdot]$ at R_N is the set of undominated alternatives. It is denoted $C(E[\cdot], R_N)$. $E[\cdot]$ is *stable* if $C(E[\cdot], R_N) \neq \emptyset$ for all $R_N \in L(A)^N$.

In order to formulate the notion of cycle it is interesting to introduce the following notation : For $U \in \mathcal{P}_0(A)$ and $S \in \mathcal{P}(N)$ put :

$$E[U](S) = \bigcap_{a \in U} E[a](S) \tag{7}$$

Definition 2.11 A *cycle* in $E[\cdot]$ is an r-tuple $((U_1, B_1, S_1), \ldots, (U_r, B_r, S_r))$ where $r \geq 1$ $U_k \in \mathcal{P}_0(A)$, $B_k \in E[U_k](S_k)$ $(k = 1, \ldots, r)$ and such that :

(i) $\cup_{k=1}^{r} U_k = A$

(ii) For any $\emptyset \neq J \subset \{1, \ldots, r\}$ such that $\cap_{k \in J} S_k \neq \emptyset$ there exists $k \in J$ such that for all $l \in J$, $U_k \cap B_l = \emptyset$

The *length* of the cycle is the natural number r.

A cycle is *strict* if $(U_1, \ldots, U_r)$ is a partition of A. $E[\cdot]$ is *acyclic* if it has no cycle.

The length of a cycle, the stability index denoted $\sigma(E[\cdot])$ and the theorem 3.4 stating the equivalence between acyclicity and stability extend *verbatim* to the local effectivity function.

There is a mapping $E_\xi : \mathcal{P}(N) \to \mathcal{P}(\mathcal{P}_0(A)$ which plays a special role in the study of stability. It is defined by:

$$E_\xi(S) = \{B \in \mathcal{P}_0(A) \mid B = A \,\text{or}\, \exists a \notin B, B \in E[a](S)\} \tag{8}$$

Note that properties (i) and (ii) of definition 2.3 are satisfied by E_ξ so that the latter would have been an effectivity function where it monotonic. Nevertheless one has:

$$\{B \in \mathcal{P}_0(A) \mid \forall B' \supset B, B' \in E_\xi(S)\} = E[A](S) \tag{9}$$

The mapping $E[A] : P(N \to \mathcal{P}(P_0(A)$ satisfies properties (i) -(iii) of definition 2.3 so that it is an effectivity function. This is the (global) effectivity function induced by the local effectivity function $E[\cdot]$. We shall denoted it E_β. Clealy one always have:

$$C(E[\cdot], R_N) \subset C(E_\beta, R_N) \tag{10}$$

Lemma 2.12 *$E_\xi = E_\beta$ if and only if E_ξ is monotonic (property (iii) of definition 2.3), or equivalently if and only if E_ξ is an effectivity function.*

Clearly any cycle in E_β is a cycle in $E[\cdot]$. Moreover a 2-cycle in $E[\cdot]$, though not necessarily a cycle in $\mathcal{E}_\beta$, gives rise to a 2-cycle in E_β.

Lemma 2.13 $\sigma(E[\cdot]) = 2$ *if and only if* $\sigma(E_\beta) = 2$

The last property does not extend to longer cycles. However one has:

Lemma 2.14 *(i) If* $E_\xi = E_\beta$ *then any cycle in* $E[\cdot]$ *is a cycle in* E_β

Lemma 2.15 *Assume that* E_β *is maximal and regular then:*
(i) $E_\xi = E_\beta$ *if and only if any cycle in* $E[\cdot]$ *is a cycle in* E_β*. Precisely:*
(ii) If $E_\xi \neq E_\beta$ *then there exists a cycle of length 3 in* $E[\cdot]$ *that is not a cycle in* E_β

Theorem 2.16 *Assume that* E_β *is maximal and regular then:* $E_\xi = E_\beta$ *if and only if* $C(E_\beta, R_N) = C(E[\cdot], R_N)$ *for all* $R_N \in L(A)^N$

Theorem 2.17 *Assume that* E_β *is maximal and regular then:*
(i) If $\sigma(E[\cdot]) = +\infty$ *then* $\sigma(E_\beta) = +\infty$
(ii) If $\sigma(E[\cdot]) < +\infty$ *and* $\sigma(E_\beta) = +\infty$ *then* $\sigma(E[\cdot]) = 3$
(iii) If $\sigma(E[\cdot]) < +\infty$ *and* $\sigma(E_\beta) < +\infty$ *then* $\sigma(E_\beta) = \sigma(E[\cdot]) = 3$

3 Interactive forms

In this section we present a general model of interaction. The elements of A are viewed as social situations or states. At any state $a \in A$ we dispose of a description of the acting power of the agents in the society. The acting power which depends generally on a is represented by a set of interaction arrays. Thus if the state of the society is a, some individuals or coalitions can move or threat to move to other states upsetting therefore the state a. Formally we define the following:

An *interaction array* on (N, A) is a mapping $\varphi : \mathcal{P}(N) \to \mathcal{P}(A)$. Let $\Phi = \Phi(N, A)$ be the set of all interaction arrays. We endow $\Phi(N, A)$ with the partial order $\leq$ where $\varphi \leq \varphi'$ if and only if $\varphi(S) \subset \varphi'(S)$ for all $S \in \mathcal{P}(N)$. The support of φ denoted $[\varphi]$ is the set $S \in \mathcal{P}(N)$ such that $\varphi(S) \neq \emptyset$.

Definition 3.1 An *interactive form* over (N, A) is a mapping $\mathcal{E}$ from $\mathcal{P}_0(A)$ to subsets of $\Phi(N, A)$ satisfying the following conditions:
(i)$\varphi \in \mathcal{E}[U] \Rightarrow [\varphi] \neq \emptyset$
(ii) $\varphi \leq \varphi', \varphi \in \mathcal{E}[U] \Rightarrow \varphi' \in \mathcal{E}[U]$

An interactive form is an *interactive presheaf* if :

$$U \subset V \Rightarrow \mathcal{E}[V] \subset \mathcal{E}[U] \tag{11}$$

An interactive presheaf $\mathcal{E}$ is a *sheaf* if for any set U and any collection $(U_i, i \in I)$ such that $U = \cup_{i \in I} U_i$, one has

$$\mathcal{E}[U] = \bigcap_{i \in I} \mathcal{E}[U_i] \tag{12}$$

Clearly $\mathcal{E}$ is a sheaf if and only if : $E[U] = \cap_{a \in U} \mathcal{E}[a]$ An interactive sheaf is completely detemined by the family: $(\mathcal{E}[a], a \in A)$. Any such a family that satisfies relations similar to those of definition (3.1) will be called an *interactive bundle* and the interactive sheaf that it induces is defined by:

$$\mathcal{E}[U] = \bigcap_{a \in A} \mathcal{E}[a] \tag{13}$$

We may think of an interaction array in $\mathcal{E}[U]$ as a description of the possible joint action of the agents given any state in U. The natural interpretation of the statement $\varphi \in \mathcal{E}(U)$, and indeed the one which will be used in the sequel, is as follows: When the outcome or state is in U, then some coalition S can move the outcome into the set $\varphi(S)$. Notice that this interpretation is of the β- type in other words it is defensive or reactive type and not of forcing type (α- type). All players and coalitions can react to a proposal in U, that is to say that any coalition can act at a by threatening to move the outcome into a new state but at least one coalition S can move it into $\varphi(S)$. Whether that coalition has a real interest to make this threat depends on the actual preferences. This is why we introduce the following:

Definition 3.2 An alternative a is *dominated* at the preference profile R_N if there exists some $U \in \mathcal{P}_0(A)$, and some $\varphi \in \mathcal{E}(U)$ such that $\varphi(\emptyset) = \emptyset$ and $\varphi(S) \subset P(a, S, R_N)$ for all $S \in \mathcal{P}_0(N)$. The alternative a is a *settlement* at R_N if it is not dominated at R_N. The set of all settelments at R_N will be denoted: $SET(\mathcal{E}, R_N)$.

We shall use the following notations. For $\varphi \in \Phi(N, A))$ we define the range of φ as $R(\varphi) = \bigcup_{S \in \mathcal{P}_0(N)} \varphi(S)$. For $i \in N$ we put $R^i(\varphi) = \bigcup_{S \ni i} \varphi(S)$. Let $\mathcal{E}[\cdot]$ be an interactive form.

Definition 3.3 A cycle in $\mathcal{E}$ is an r-tuple $(U_1, V_1, \varphi_1) \ldots (U_r, V_r, \varphi_r)$ where: $U_k \in \mathcal{P}_0(A)$,$V_k \in \mathcal{P}_0(A)$, $U_k \subset V_k$ $\varphi_k \in \mathcal{E}[V_k]$ $k(= 1, \ldots, r)$ with the properties:

(i) $\cup_{k=1}^{r} U_k = A$,
(ii) if $i \in N$ and $\emptyset \neq J \subset \{1, \ldots, r\}$ then there exists $k \in J$ such that for all $l \in J$, $U_k \cap R^i(\varphi_l) = \emptyset$.
The natural number r is the *length* of the cycle. Such a cycle will be called an r- cycle.
A cycle is said to be *strict* if in (i) $(U_1, \ldots, U_r)$ is a partition of A.

The interactive form $\mathcal{E}$ is said to be *acyclic* (*strictly acyclic*) if there are no cycles (strict cycles) in $\mathcal{E}$.

In almost the same way as is Abdou - Keiding [6] we have:

Theorem 3.4 *$\mathcal{E}$ is stable if and only if $\mathcal{E}$ is acyclic.*

In order to measure the degree of instability when this occur, we introduce:

Definition 3.5 The *stability index* of $\mathcal{E}$, denoted $\sigma(\mathcal{E})$, is the minimal length of a cycle in $\mathcal{E}$. This number is set to $+\infty$ if $\mathcal{E}$ is acyclic.

3.1 Confederation and Confederation Structures

Now it may be the case that institutionnally (by law or physical impossibilities) some coalitions are not allowed to form, that is only coalitions in some $\mathcal{S} \subset \mathcal{P}_0(N)$ can actually be active. We call *confederation* any $\mathcal{S} \subset \mathcal{P}(N)$. The definitions can be adapted in consequence. An alternative a is $\mathcal{S}$-*dominated* at the preference profile R_N if there exists some $U \in \mathcal{P}_0(A)$, and some $\varphi \in \mathcal{E}(U)$ such that $[\varphi] \subset \mathcal{S}$, $\varphi(S) \subset P(a, S, R_N)$ for all $S \in \mathcal{S} \setminus \{\emptyset\}$. The alternative a is an $\mathcal{S}$ *-settlement* at R_N if it is not $\mathcal{S}$-dominated at R_N. The set of all $\mathcal{S}$-settlements at R_N will be denoted: $SET(\mathcal{E}, \mathcal{S}, R_N)$.

The restriction of $\mathcal{E}$ to $\mathcal{S}$ is the interactive form $\mathcal{E}(\mathcal{S})$ defined by :

$$\mathcal{E}(\mathcal{S})[U] = \{\varphi \in \Phi \mid \exists \varphi' \in \mathcal{E}[U], [\varphi'] \subset \mathcal{S}\} \tag{14}$$

Clearly one has: $SET(\mathcal{E}, \mathcal{S}, R_N) = SET(\mathcal{E}(\mathcal{S}), R_N)$. Furthermore, any nonempty set $\mathfrak{F}$ of confederations will be called a *confederation structure.* The restriction of $\mathcal{E}$ to $\mathfrak{F}$ is defined by:

$$\mathcal{E}_{\mathfrak{F}}[U] = \{\varphi \in \mathcal{E}[U] \mid \varphi' \in \mathcal{E}[U], \varphi' \leq \varphi \text{ and } \exists \mathcal{S} \in \mathfrak{F}, [\varphi'] \subset \mathcal{S}\} \tag{15}$$

The notions of $\mathfrak{F}$- settlement and $\mathfrak{F}$-stability are defined in consequence: An alternative a is $\mathfrak{F}$-*dominated* at the preference profile R_N if there exists some $\mathcal{S} \in \mathfrak{F}$ and some $U \in \mathcal{P}_0(A)$ and $\varphi \in \mathcal{E}(\mathcal{S})[U]$ such that $[\varphi] \subset \mathcal{S}$, $\varphi(S) \subset P(a, S, R_N)$ for all $S \in \mathcal{S} \setminus \{\emptyset\}$. The alternative a is an $\mathfrak{F}$ *-settlement* at R_N if it is not $\mathfrak{F}$-dominated at R_N. The set of $\mathfrak{F}$ -settlements at R_N will be denoted: $SET(\mathcal{E}, \mathfrak{F}, R_N)$. It is clear from the definition that:

$$SET(\mathcal{E}, \mathfrak{F}, R_N) = SET(\mathcal{E}_{\mathfrak{F}}, R_N) = \cap_{\mathcal{S} \in \mathfrak{F}} SET(\mathcal{E}, \mathcal{S}, R_N)$$

Moreover given a confederation $\mathcal{S}$ one can associate to it any one of the confederation structures $\mathfrak{F}$ that contains (necessarily) $\mathcal{S}$ and possibly some subsets of $\mathcal{S}$. Obviously $SET(\mathcal{E}, \mathfrak{F}, R_N) = SET(\mathcal{E}, \mathcal{S}, R_N)$.

An interactive form $\mathcal{E}$ (resp. presheaf, sheaf) is an *effectivity form* (resp. *presheaf, sheaf*) if the following properties hold:

(i) If $[\varphi] = \{\emptyset\}$ then $\varphi \in \mathcal{E}[U]$ if and only if $U \subset \varphi(\emptyset)$

(ii) $S \in \mathcal{P}_0(N), U \in \mathcal{P}_0(A), U \subset \varphi(S) \Rightarrow \varphi \in \mathcal{E}[U]$

To any local effectivity function $E[\cdot]$ one can associate "canonically" an interactive form as follows:

$$\mathcal{E}[U] = \{\varphi \in \Phi | \exists S \in \mathcal{P}(N) : \varphi(S) \in E[U](S)\} \tag{16}$$

$\mathcal{E}$ is actually an effectivity form. Reciprocally to any affectivity form $\mathcal{E}$, the restriction of $\mathcal{E}$ to the confederation structure $\mathfrak{F} = \{\{S\} \mid S \in \mathcal{P}(N)\}$, $\mathcal{E}_{\mathfrak{F}}$ "is" the local effectivity function induced by $\mathcal{E}$. In fact we shall put:

$$E[U](S) = \{B \in P_0(N) \mid \exists \varphi \in \mathcal{E}[U], [\varphi] = \{S\}, \varphi(S) = B\} \tag{17}$$

The induced effectivity function is defined by:

$$E_\beta(S) = E[A](S) \tag{18}$$

4 Stratified effectivity forms

We recall that a subset of coalitons $\mathcal{S} \in \mathcal{P}(N)$ is called a confederation. The dual of $\mathcal{S}$ is the confederation defined by $\mathcal{S}^* = \{S \in \mathcal{P}(N) \mid S^c \in \mathcal{S}\}$. A confederation $\mathcal{S}$ is said to be *admissible* if $\mathcal{S} = \{N\}$ or if the elements of $\mathcal{S}^*$ are non-empty and disjoint. The set of all admissible confederations is a particular confederation structure that will be denoted $\mathfrak{P}$. An effectivity bundle $(\mathcal{E}[a], a \in A)$ is said to be *stratified* if any $\varphi \in \mathcal{E}[a]$ has its support in some admissible confederation. In this section we assume that we are given a stratified effectivity bundle and its associated sheaf. For every integer $r \geq 1$, let $\mathfrak{P}_r$ be the set of $\mathcal{S} \in \mathfrak{P}$ with $|\mathcal{S}| \leq r$, and let $\mathcal{E}_r$ be the interactive form that contains the elements $\varphi \in \mathcal{E}$ with support in $\mathfrak{P}_r$, that is $\mathcal{E}_r = \mathcal{E}_{\mathfrak{P}_r}$. Given that the cardinal of A is n, it follows that:

$$\{N\} \subset \mathfrak{P}_1 \subset \mathfrak{P}_2 \subset \cdots \subset \mathfrak{P}_n = \mathfrak{P}_{n+1} = \cdots = \mathfrak{P} \tag{19}$$

so that $\mathcal{E}_n = \mathcal{E}$ and :

$$C(E_\beta, R_N) \supset SET(\mathcal{E}_1, R_N) \supset \cdots \supset SET(\mathcal{E}_n, R_N) = SET(\mathcal{E}, R_N) \tag{20}$$

Recall that the range of φ is defined by $R(\varphi) = \bigcup_{S \in \mathcal{P}_0(N)} \varphi(S)$
For any $\mathcal{S} \in \mathfrak{P}$ we define the following sets:

$$E_\beta(\mathcal{S}) = \{\varphi \in \Phi |\ [\varphi] \subset \mathcal{S}, \text{and } \exists S \in \mathcal{S}, \varphi(S) \in E_\beta(S)\} \tag{21}$$

$$E_\xi(\mathcal{S}) = \{\varphi \in \Phi |\ [\varphi] \subset \mathcal{S}, \text{and either } R(\varphi) = A \text{ or } \exists a \notin R(\varphi), \varphi \in \mathcal{E}[a]\} \tag{22}$$

$$D(\mathcal{S}) = \{\varphi \in \Phi |\ [\varphi] \subset \mathcal{S}, S \neq T \Rightarrow \varphi(S) \cap \varphi(T) = \emptyset\} \tag{23}$$

Definition 4.1 Let $r \geq 1$. $\mathcal{E}$ is r *-exact* if for all $\mathcal{S} \in \mathfrak{P}_r$ one has:

$$E_\xi(\mathcal{S}) \cap D(\mathcal{S}) = E_\beta(\mathcal{S}) \cap D(\mathcal{S}) \tag{24}$$

Lemma 4.2 *Assume E_β is maximal and stable. If $\mathcal{E}$ is not r- exact, then there exists $a \in A$ and $R_N \in L(A)^N$ such that $SET(\mathcal{E}_r, R_N) = \emptyset$ and $C_\beta(E_\beta, R_N) = \{a\}$*

Theorem 4.3 *Assume E_β is maximal and stable. Then $SET(\mathcal{E}_r, \cdot) = C(E_\beta, \cdot)$ if and only if $\mathcal{E}$ is r- exact*

Theorem 4.4 *Assume E_β is maximal. Then the following are equivalent:*
(i) $\mathcal{E}_r$ is stable
(ii) E_β is stable and r- exact
(iii) E_β is superadditive and subadditive and $\mathcal{E}$ is r- exact
and in this case $SET(\mathcal{E}_r, R_N) = C(E_\beta, R_N)$ for all $R_N \in L(A)^N$

Therefore we obtain the following localization of the index in case of instability:

Theorem 4.5 *Assume E_β is maximal :*
(i) If E_β is not stable then $\sigma(\mathcal{E}) \leq 3$
(ii) If E_β is stable but If $\mathcal{E}$ is not r-exact then $\sigma(\mathcal{E}) \leq r + 2$.

5 Stability index of strategic game forms

We consider a *game form* $G = (X_1, \ldots, X_n, A, g)$ where X_i is the *strategy set* of player i, $(i \in N)$ and $g : \prod_{i \in N} X_i \longrightarrow A$ is the *outcome function.* We shall assume that g is onto. For every coalition $S \in \mathcal{P}_0(N)$, the product $\prod_{i \in S} X_i$ is denoted X_S (by convention $X_\emptyset$ is the singleton $\{\emptyset\}$) and $N \backslash S$ is denoted S^c. Similarly if $B \in \mathcal{P}(A)$, $A \backslash B$ is denoted B^c. If $x_N \in X_N$, the notation $g(x_S, X_{S^c})$ stands for $\{g(x_S, y_{S^c}) \mid y_{S^c} \in X_{S^c}\}$ if $S \neq \emptyset$ and for $g(X_N)$ if $S = \emptyset$. A game in strategic form is an array $(X_1, \ldots, X_n; Q_1, \ldots, Q_n)$, where for each $i \in N = \{1, \ldots, n\}$ $\hat{X}_i$ is a non-empty set of strategies of player i, and Q_i is a quasi-order (complete, transitive, reflexive binary relation) on $X_N = \prod_{i \in N} X_i$. For each preference profile $R_N \in L(A^N)$, the game form G induces a game $(X_1, \ldots, X_n; Q_1, \ldots, Q_n)$ with the same strategy spaces as in G and with the Q_i defined by

$$x_N \, Q_i \, y_N \Leftrightarrow g(x_N) \, R_i \, g(y_N)$$

for $x_N, y_N \in X_N$. We denote this game by (G, R_N).

Let $\mathcal{S}$ be a confederation. For $R_N \in Q(A)^N$ a preference profile, a strategy array $x_N \in X_N$ is an $\mathcal{S}$*-equilibrium* of the game (G, R_N) if there is no coalition $S \in \mathcal{S}$ and $y_S \in X_S$ such that

$$g(y_S, x_{S^c}) \in P(g(x_N), S, R_N).$$

We say that $a \in A$ is an $\mathcal{S}$*-equilibrium outcome* of (G, R_N) if there is an $\mathcal{S}$-equilibrium $x_N \in X_N$ with $g(x_N) = a$. The game form G is said to be *solvable in* $\mathcal{S}$*-equilibrium* or $\mathcal{S}$*-solvable*, if for each preference profile $R_N \in Q(A)^N$, the game (G, R_N) has an $\mathcal{S}$-equilibrium. In particular, when $\mathcal{S} = \mathcal{N} = \{\{1\}, \ldots, \{n\}\}$, an $\mathcal{S}$-equilibrium is simply a Nash equilibrium. Similarly, when $\mathcal{S} = \mathcal{P}_0(N)$, an $\mathcal{S}$-equilibrium is a strong (Nash) equilibrium. In what follows we assume that all profilies are in $L(A)^N$.

Given the game form $G = (X_1, \ldots, X_n, A, g)$ the β*-effectivity form* (over (N, A)) associated with G is the interactive form $\mathcal{E}_\beta^G$ defined as follows: For $U \in \mathcal{P}_0(A)$:

$\mathcal{E}_\beta^G[U] =$

$$\{\varphi \in \Phi(N, A) \mid \forall y_N \in g^{-1}(U), \exists S \in \mathcal{P}(N), \exists x_S \in X_S : g(x_S, y_{S^c}) \in \varphi(S)\} \quad (25)$$

It is easy to verify that $\mathcal{E}_\beta^G$ is actually an effectivity sheaf Let $\mathcal{E}_\beta^G(\mathcal{S})$ be restriction of $\mathcal{E}_\beta$ to the confederation $\mathcal{S}$ as was done in (15).

Lemma 5.1 *Let $G = (X_1, \dots, X_n, A, g)$ be a game form. The set of $\mathcal{S}$-equilibrium outcomes of (G, R_N) coincides with the settlement set of $\mathcal{E}_\beta^G(\mathcal{S})$ at R_N. Therefore G is $\mathcal{S}$- solvable if and only if $\mathcal{E}_\beta^G(\mathcal{S})$ is stable.*

Define the stability index of $(G, \mathcal{S})$ as the stability index of $\mathcal{E}_\beta^G(\mathcal{S})$ It will be denoted $\sigma(G, \mathcal{S})$

Theorem 5.2 *The stability index of a two -player game form in Nash equilibrium is either* 2 *or* $+\infty$

This is a corollary of the fact that for these game forms Nash solvability is equivalent to tightness. See Gurvich [8], [10] or Abdou [2]. If we consider the class of rectangular game forms (see Gurvich [9] and Abdou [3]) one hase a similar characterization for Nash solvability by tightness. It does not follow however that the Nash-Stability index of such a game form is 2 in case of instabily. The only thing that is obvious is that the core index of such a game form is 2

Theorem 5.3 *(i) The stability index of a two -player game form in strong Nash equilibrium is either* $2, 3, 4$ *or* $+\infty$.

(ii) If the game form is tight then we have the alternative: either the index $\in \{3, 4\}$ *this is the case if and only if the game form is not it* 2 *-exact or the index is* $+\infty$ *and this is the case if and only if it is* 2*-exact*

This is a corollary of a theorem by Abdou [2] characterizing strong solvability of two-player game forms

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